Semi Anti-Integral Elements and Integral Extensions

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Let R be a Noetherian domain with quotient field K. Let L be an extension of a field K and let α be an element of L which is algebraic over K. Put $d = [K[\alpha] : K]$. We consider the canonical exact sequence

$$0 \longrightarrow I \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

If I is generated by polynomials of degree d, then α is said to be the anti-integral element over R. Our unexplained technical terms are standard and are seen in [1] and [2]. Let $A = R[\alpha]$. It is well known that if α is integral and anti-integral over R, then A is a free R-module of rank d([3].)

Note that α has an unique monic relation of degree d over K. Let $\phi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, $\eta_i \in K$, be a monic relation of α over K and let $I_{\eta_i} = R :_R \eta_i = \{a \in R | a\eta_i \in R\}$ and $I_\alpha = \bigcap_i I_{\eta_i}$. Then we have :

PROPOSITION 1. The following statements are equivalent to each other ;
(1) α is anti-integral over R,
(2) $R[X] :_{K[X]} \phi_\alpha(X) \subseteq I_\alpha R[X]$.

PROOF. (1)⇒(2) : For any element g(X) of $R(X) :_{K[X]} \phi_\alpha(X)$, we have $g(X)\phi_\alpha(X) = f(X)$ for some $f(X) \in R[X]$. Let $\rho : R[X] \longrightarrow R[\alpha]$ be a natural ring homomorphism. Then $f(x) \in \mathrm{Ker}\ \rho$. We consider the natural exact sequence

$$0 \longrightarrow \mathrm{Ker}\ \rho \longrightarrow R[X] \xrightarrow{\rho} R[\alpha] \longrightarrow 0.$$

Since α is anti-integral over R, $\mathrm{Ker}\ \rho = I_\alpha \phi_\alpha(X) R[X]$, (see[3]) and hence we have $f(X) = \sum_i (h_i \phi_\alpha(X)) t_i(X)$ for some $h_i \in I_\alpha$, $t_i(X) \in R[X]$. Cancelling $\phi_\alpha(X)$, we see that $g(X) = \sum_i h_i t_i(X) \in I_\alpha R[X]$.
(2)⇒(1) : For any element $f(X) \in \mathrm{Ker}\ \rho$, we have $f(X) = g(X)\phi_\alpha(X)$ for some $g(X) \in K[X]$. Thus $g(X) \in R[X] :_{K[X]} \phi_\alpha(X) \subseteq I_\alpha R[X]$ by assumption. Hence we can write $g(X) = \sum_i h_i t_i(X)$, $h_i \in I_\alpha$, $t_i(X) \in R[X]$. Therefore we obtain $f(X) = \sum_i (h_i \phi_\alpha(X)) t_i(X) \in I_\alpha \phi_\alpha(X) R[X]$. Thus α is anti-integral R.

Q.E.D.

This observation leads to the following definition.

DEFINITION 2. If $R[X] :_{K[X]} \phi_\alpha(X) \subseteq R[X]$, α is said to be a semi anti-integral element over R.

If α is anti-integral element over R, it is semi anti-integral element over R by Proposition 1.

REMARK 3. Let $\bar{R}$ be the integral closure of R in K. By Gauss's lemma, we have easily seen $\eta_i \in \bar{R}$ for all i, if α is integral over R.

Next we shall show that if α is an integral and semi anti-integral element over R, then α is an anti-integral element over R. For the purpose of this proof, we need the following Proposition. If α is an anti-integral element over R, we put $\mathbf{H}_\alpha = R[X] :_{K[X]} \phi_\alpha(X)$.

PROPOSITION 4. The following statements are equivalent to each other ;

(1) α is anti-integral over R,

(2) $\mathbf{H}_\alpha = I_\alpha R[X]$.

PROOF. (1)⇒(2) : Let g(X) be an arbitrary element of $\mathbf{H}_\alpha$. Then, from the proof (1)⇒(2) of Proposition 1, we can write $g(X) = \sum_i h_i t_i(X)$, $h_i \in I_\alpha$, $t_i(X) \in R[X]$. Hence we see that $g(X) \in I_\alpha R[X]$. Since one inclusion is obvious, we have desired result.

(2)⇒(1) : Assume that $f(\alpha) = 0$ for some $f(X) \in R[X]$. Then we have $f(X) = g(X)\phi_\alpha(X)$ for some $g(X) \in K[X]$. Thus $g(X) \in R[X] :_{K[X]} \phi_\alpha(X) = \mathbf{H}_\alpha$. Hence we have $g(X) = \sum_i g_i X^i$, $g_i \in I_\alpha$. Note that $g_i\, \phi_\alpha(X) \in R[X]$ and $\deg(g_i\, \phi_\alpha(X)) = d$ for all i. Therefore f(X) is generated by $I_\alpha \phi_\alpha(X)$. Thus α is anti-integral over R.

Q.E.D.

As a consequence, we have :

THEOREM 5. If α is an integral and semi anti-integral element over R, then α is an anti-integral element over R, and hence $A = R[\alpha]$ is a free R-module of rank d.

PROOF. Since α is integral over R, there exsists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. First, we consider the case of deg f(X) = d. In this case, α is anti-integral over R, and hence it is nothing to prove. Next we assume that $n = \deg f(X) > d$. Let

$$\begin{aligned} f(X) &= (X^{n-d} + \zeta_1 X^{n-d-1} + \cdots + \zeta_{n-d})\phi_\alpha(X) \\ &= (X^{n-d} + \zeta_1 X^{n-d-1} + \cdots + \zeta_{n-d})(X^d + \eta_1 X^{d-1} + \cdots + \eta_d), \end{aligned}$$

ζ_i, $\eta_i \in K$. Since α is semi anti-integral over R, we see $\zeta_1, \zeta_2, \ldots, \zeta_{n-d} \in R$ by the definition. If we equate coefficients of X^{n-1}, we see $\eta_1 + \zeta_1 \in R$, and hence $\eta_1 \in R$.

Proceeding in this way, we get $\eta_i \in R$ for all i. Thus $\phi_\alpha(X) \in R[X]$, so that α is anti-integral over R.

Q.E.D.

REMARK 6. Let R be a Noetherian domain with quotient field K. Let L be an extension a field K and Let α be an element of L which is algebraic over K. Put $d = [K[\alpha]:K]$. Assume that α is integral over R. Then there exsists a monic polynomial $f(X) \in R[X]$ such that $f(\alpha) = 0$. Then we have ;

$$\deg f(X) = d \Leftrightarrow A \text{ is a free R-module of rank } d \Leftrightarrow \phi_\alpha(X) \in R[X] \Leftrightarrow f(X) = \phi_\alpha(X)$$

Thus we obtain the following Proposition. we omit the proof.

PROPOSITION 7. $\{p \in Spec(R) | A_p \text{ is not flat over } R_p\} = \{p \in Spec(R) | p \supseteq I_\alpha\}$.

We assume that there exsists a monic polynomial $f(X) \in R[X]$ of degree $d+1$ such that $f(\alpha) = 0$. Then we have $f(X) = (X+\zeta)\phi_\alpha(X)$, $\zeta \in K$. It follows that $\zeta \in \bar{R}$. Thus,

PROPOSITION 8. The following statements are equivalent to each other.

(1) $\zeta \in R$

(2) $\phi_\alpha(X) \in R[X]$.

Therefore we have $\{p \in Spec(R) | A_p \text{ is not flat over } R_p\} = \{p \in Spec(R) | p \supseteq I_\zeta\}$.

PROOF. Let $f(X) = X^{d+1} + a_1X^d + \cdots + a_d$ and $\phi_\alpha(X) = X^d + \eta_1X^{d-1} + \cdots + \eta_d$. Since $f(X) = (X+\zeta)\phi_\alpha(X)$, we deduce that $a_1 = \zeta + \eta_1$, $a_i = \zeta\eta_{i-1} + \eta_i (2 \leqq i \leqq d)$ and $a_{d+1} = \zeta\eta_d$. Note that $a_i \in R (1 \leqq i \leqq d)$. Thus we see thet $\zeta \in R$ is equivalent to $\eta_1 \in R$. The second half is easily seen the fact that A_p is flat over R_p if and only if $\phi_\alpha(X) \in R_p[X]$. This complete the proof.

Q.E.D.

REMARK 9. Consider the canonical exact sequence

$$0 \longrightarrow P \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0.$$

Under the above condition, we have $P = I_\alpha\phi_\alpha(X)R[X] + f(X)R[X]$.

Finally, we have the following :

THEOREM 10. Let R be a Noetherial domain with quotient field K. Let L be an extension of a field K and let α be an element of L which is algebraic over K. Let $d = [K[\alpha]:K]$ and $\phi_\alpha(X) = X^d + \eta_1X^{d-1} + \cdots + \eta_d$, $\eta_i \in K$ be a monic relation of α over K. Assume that α is integral over R. Let B be a intermediate ring between R and $\bar{R}$. Then the following statements are equivalent to each other ;

(1) $R[X] :_{K[X]} \phi_\alpha(X) \subseteq B[X]$,

(2) $B[\alpha]$ is flat over B and α is anti-integral over B.

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