Boundary Matrix Method for Eigenvalue Problems

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Abstract

The boundary matrix method for solving eigenvalue problems for the Laplace operator is formulated in this paper. The numerical methods are based on a nonlinear representation of the eigenvalue problem on the boundary. The nonlinear eigenvalue problems are solved by using the Newton iteration method. Numerical examples for simple models by the present methods are shown. From the numerical solutions the present methods give us accurate numerical eigen-modes even for high eigenfrequencies.

1. INTRODUCTION

Numerical methods for solving the eigenvalue problem in the form of

$$-\triangle u(x) = \lambda^2 u(x) \ in \ \Omega \tag{1}$$

where Ω denotes a bounded region in R^k ($k = 1, 2, 3$), with the boundary condition :

$$u(x) = g(x) \ on \ \partial\Omega \tag{2}$$

have been studied by using the finite difference method, the finite element method and the boundary element method, where $\partial\Omega$ denotes the boundary of the domain Ω, If we seek approximate solutions of high eigenfrequencies with the finite difference and finite element methods it is necessary to take a fine mesh and element discretization. If the sizes of the finite diference mesh and the finite element are not enough small to approximate the eigenfunction of the problem, the ghost solution (the inaccurate solution) is occurred. In order to avoid the difficulty the Petrov Galerkin finite element method was presented [1, 2, 3]. Appling those methods to the problem generate linear algebraic eigenvalue problems. The boundary element approach is different from those methods since it is necessary to use the fundamental solution of a differential operator to formulate the boundary integral equation. If we take the fundamental solution for the Laplace operator we obtain an integral equation formulation. In this case we also obtain the linear algebraic eigenvalue problem [5]. The searching method with boundary integral equation method with the fundamental solution for the Helmholtz operator was presented by Niwa et al. [4]. For plate problems with the boundary integral equation method we refer to the text book by Kitahara [5]. The searching method gives numerical eigen frequencies which satisfy the determinat free condition of the matrix generated by the boundary element discretization. Since the variation of the determinant near eigenvalues is steep and sensitive it is difficult the determin eigenvalues accurately. On the other hand the boundary element approach the two advantages :

1. It is unnecessary to discretize the interior of the given domain.
2. When we use fundamental solutions of the Helmholtz operators, we can avoid ghost modes, since numerical eigenfunctions are expressed with the fundamental solution with the eigenparameter.

The author presents numerical methods by using the fundamental solution of Helmholtz operator. For one-dimensional problems the boundary matrix method is formulated. Taking account of the normalizing condition of wight coefficients for approximation, nonlinear algebraic eigenvalue problems on the given boundary are induced by appling the present methods. By using the Newton we obtain numerical solutions. From numerical experiments we show that the present methods give accurate numerical solutions with small unknowns. Moreover it is shown that the boundary matrix method has approximately uniform accuracy with respect to the frequency of vibration.

2. BOUNDARY MATRIX METHOD

The boundary matrix method to solve a one-dimensional eigenvalue problem is formulated in this section. When we consider a one-dimensional boundary value problem the approach which is similar to the boundary integral equation method does not generate an integral equation, since the boundary of the region is consisted with two points. For the one-dimensional case we obtain a 2 by 2 linear system by using the boundary integral equation method. Therefore we call the approach for one-dimensional problems the boundary matrix method. Let us formulate the boundary matrix method for the eigenvalue problem :

$$-\frac{d^2u(x)}{dx^2}-\lambda^2u(x)=0 \qquad in\ (0,1) \tag{3}$$

The linear combination of the fundamental solutions with some source points gives us a solution of the equation (3). For the one-dimensional case the fundamental solution is

$$\frac{1}{2\lambda}\exp(-i\lambda r_1), \tag{4}$$

where x, y and $r_1=|x-y|$ are the observation point, the sourse point and the distance between them, respectively. If we consider the Dirichlet boundary condition : $u(0)=$

$u(1) = 0$, the matrix equation is

$$\begin{bmatrix} \frac{1}{2\lambda} & \frac{1}{2\lambda}\exp(-i\lambda) \\ \frac{1}{2\lambda}\exp(-i\lambda) & \frac{1}{2\lambda} \end{bmatrix} \begin{Bmatrix} c_0 \\ c_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

where c_0 and c_1 are the weight coefficients for the linear combination of fundamental solutions at two points. Adding the following normalizing equation of those weight coefficients :

$$c_0^2 + c_1^2 = 1 \tag{6}$$

we obtain the nonlinear system (5)-(6) including three unknowns λ, c_0 and c_1. Each term in the nonlinear system has its derivative on each unknown. Therefore it is possible to apply the Newton iteration method to the system. The iteration procedure is

$$\begin{Bmatrix} c_0^m+1 \\ c_1^m+1 \\ \lambda^m+1 \end{Bmatrix} = \begin{Bmatrix} c_0^m \\ c_1^m \\ \lambda^m \end{Bmatrix} - J_m^{-1} \begin{Bmatrix} c_0^m \\ c_1^m \\ \lambda^m \end{Bmatrix} \tag{7}$$

where J_m is the Jacobian matrix of the nonlinear system, as follows :

$$J_m = \begin{bmatrix} 1 & \frac{1}{2\lambda^m}\exp(-i\lambda^m) & -\left(\frac{\exp(-i\lambda^m)}{2\lambda^2} + i\frac{\exp(-i\lambda^m)}{2\lambda}\right)c_1^m \\ \frac{1}{2\lambda^m}\exp(-i\lambda^m) & 1 & -\left(\frac{\exp(-i\lambda^m)}{2\lambda^2} + i\frac{\exp(-i\lambda^m)}{2\lambda}\right)c_0^m \\ 2c_0^m & 2c_0^m & 0 \end{bmatrix} \tag{8}$$

4. NUMERICAL EXPERIMENTS

Numerical results for one-dimensional problems and a two-dimensional problem are shown in this section. For the equation (3) set the boundary conditions :

(P1) $u(0) = u(1) = 0$
(P2) $u(0) = 0$ and $du(x)/dx = 0$ at $x = 1$.

Numerical solutions of (P1) with the present method are shown in the table 1. For (P2) numerical solutions with the present method are shown in the table 2. Exact solutions are all real values. Numerical solutions have quite small imaginary part as calculating errors.

Table 1 Numerical solutions for (P1) with the boundary matrix method.

initial value of λ	Numerical sol	Exact	Re [sol.]/π
1	$-3.14159-2.129.3\times10^{-17}j$	$\pm\pi$	-1
10	$9.42478+2.44906\times10^{-17}j$	$\pm3\pi$	3
15	$15.708+2.70891\times10^{-15}j$	$\pm5\pi$	5.00001
20	$21.991-6.017927\times10^{-20}j$	$\pm7\pi$	6.99998
18	$28.2743+8.97138\times10^{-18}j$	$\pm9\pi$	8.99999
35	$34.5575+5.40384\times10^{-20}j$	$\pm11\pi$	11
40	$40.8407+2.46519\times10^{-32}j$	$\pm13\pi$	13
19	$298.451-7.95246\times10^{-13}j$	$\pm95\pi$	94.9999

Table 2 Numerical solutions for (P2) with the boundary matrix method.

initial value of λ	Numerical sol	Re [sol.]/π
1	$1.5708+7.42742\times10^{-14}j$	0.5
5	$4.71239+4.2825\times10^{-15}j$	1.5
9	$7.85398-6.15995\times10-16j$	2.5
10	$10.9956-2.61348\times10^{-9}j$	3.50001

5. CONCLUSION

The boundary matrix method and for solving eigenvalue problems are formulated in this paper. In the presen formulations the eigenparameter is involved nonliearly in the discretized system. Therefore the Newton method is available to calculate approximate solutions of the system. Since we apply the fundamental solution of the Helmholtz operator the discretization is carried out only on the boundary for the given domain. From numerical experiments we observe the following consequences :

1. The present methods are accurate numerical methods for solving the eigenvalue problem.
2. Accuracy of the present methods is independent of the frequency of vibration.

The second result is very important property of the present method since the result implies that we can avoid the ghost solution which is appeared in the finite difference method and the finite element method.

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