Notes on Anti-integral Elements and Pseudo-smiple Extensions

Susumu ODA* and Ken-ichi YOSHIDA**
*Uji-Yamada High School Uraguchi Ise, Mie 516 Japan
**Department of Applied Mathematics Okayama University of Science
Ridai-cho 1-1, Okayama 700, Japan

(Received September 30, 1991)

Let R be a Noetherian integral domain, let K denote the quotient field of R and let A be an R-algebra of finite type contained in K. Then A = R[F] for some fractional ideal F of R. We can write F = I α for some ideal I of R and some element α in K. So A is the form R[Iα]. In general, it seems natural to consider an extension like R[I α], where I is an ideal of R and α is an element in some extension of R, where α does not necessarily belong to K. We say that R[I α] is a pseudo-simple extension of R defined by I and α if depth $R_p > 1$ for any prime ideal P contaning I. Our objective of this paper is to study the ring like R[I α]. In particular, we are interested in the case α is anti-integral over R. The anti-integrality is studied in[3].

Throughout this paper, we use the follwing notation unless otherwise specified : let R be a Noetherial domain,

K the quotient field of R, and
R' the integral closure of R in K.

For a non-zero element α of K, put

$$R :_R \alpha := \{ a \in R : a\alpha \in R \} (= \alpha^{-1}R \cap R).$$

In this paper all rings are assumed to be commutative and to have an identity, and our general references for unexplained technical terms are[1] and[2]

We start with recalling the follwing definition which in seen in[3].

DEFINITION. *For a non-zero element α in K, we set*

$$R(\alpha) = R[\alpha] \cap R[1/\alpha]$$

*in K. We say that α is **anti-integral** over R if $R(\alpha) = R$.*

THEOREM 1. Asseume that R is a local domain with the maximal ideal **m** and let α be a non-zero element in K. Assume that α is anti-integral over R and that $R[\alpha] = R[\mathbf{m}\alpha]$. Then $\alpha \in R$ or $1/\alpha \in R$.

PROOF. Since $\alpha \in R[\mathbf{m}\alpha]$, we can write

$$\alpha = b_0 + b_1 \alpha + \ldots + b_n \alpha_n,$$

where $b_i \in \mathbf{m}^i$ for all $i > 0$ and $b_0 \in R$.

Case I: Assume that $b_0 \notin \mathbf{m}$. Then $1 - b_0(1/\alpha) = b_1 + b_2\alpha + \ldots + b_n\alpha^{n-1} \in R[\alpha] \cap R[1/\alpha] = R(\alpha) = R$. Hence $1 - b_0(1/\alpha) \in R$. Since b_0 is a unit in R, we have $1/\alpha \in R$.

Case II: Assume that $b_0 \in \mathbf{m}$. Then we have $b_0 + (b_1 - 1)\alpha + \ldots + b_n\alpha_n = 0$. Put $f(X) = b_nX^n + \ldots + b_2X^2 + (b_1 - 1)X + b_0$. Then $b_1 - 1 \notin \mathbf{m}$. Thus $C(f(X)) = R$, where $C(\ )$ denotes the content ideal. Take $h(X) \in R(X)$ which satisfies $h(\alpha) = 0$, $C(h(X)) = R$ and $\deg h(X)$ is minimal among such ones. Write

$$h(X) = c_dX_d + \ldots + c_0 \text{ with } c_i \in R \text{ and}$$
$$c_d\alpha^d + \ldots + c_0 = 0 \quad (*).$$

If c_0 is a unit in R, then by the same argument as in Case I, we have that $1/\alpha \in R$. Assume that $c_0 \in \mathbf{m}$. Then there exists $s(0 < s \leqq d)$ such that c_s is a unit in R.

From $(*)$, we have $c_d\alpha^{d-s+1} + \ldots + c_s\alpha + c_{s-1} = -(c_{s-2}\alpha^{-1} + \ldots + c_0\alpha^{-s+1}) \in R[\alpha] \cap R[1/\alpha] = R(\alpha) = R$. Thus there exists $v \in R$ such that $c_d\alpha^{d-s+1} + \ldots + c_s\alpha + v = 0$. By the minimality of $\deg h(X)$, we get $d - s + 1 = d$, that is, $s = 1$. Hence c_1 is a unit in R. Consider the equality $c_d\alpha^{d-1} + \ldots + c_1 = -(1/\alpha)c_0 \in R[\alpha] \cap R[1/\alpha] = R(\alpha) = R$. Put $t = c_d\alpha^{d-1} + \ldots + c_1 = -(1/\alpha)c_0$. If $c_1 - t \notin \mathbf{m}$, then the polynomial $c_dX^{d-1} + \ldots + c_1 - t \in R[X]$ gives a contradiction to the minimality of $\deg h(X)$. So $c_1 - t \in \mathbf{m}$. Hence $t \notin \mathbf{m}$ because c_1 is a unit in R. Thus $t = -(1/\alpha)c_0$ implies $\alpha t + c_0 = 0$ and hence $\alpha \in R$.□

COROLLARY1.1. *Assume that (R, **m**) be a normal local domain and let α be a non-zero element in K such that neither α nor $1/\alpha$ belongs to R. Then $R[\alpha] \neq R[\mathbf{m}\alpha]$.*

PROOF. Since R is normal and $R(\alpha)$ is intergral over R, $R(\alpha) = R$, that is, any $\alpha \in K$ is anti-integral over R. So our conclusion follows Theorem 1 .□

COROLLARY 1.2. *Let R be a Noetherian domain and let I be an ideal of R and let α be a non-zero element in K which is anti-integral over R. If $R[\alpha] = R[I\alpha]$, then $R_P[\alpha]$ is flat over R_P for any $P \in Spec(R)$ with $P \supset I$.*

PROOF. We may assume that R is a local domain with the maximal ideal **m.** Since $R[\alpha] = R[I\alpha]$ implies $R[\alpha] = R[\mathbf{m}\alpha]$, so that $R[\alpha]$ is flat over R by Theorem 1. □

COROLLARY 1.3. *Assume that (R, **m**) is a local domain. Then the following statements are equivalent:*

(a) *R is a DVR;*

(b) *every α in $K \setminus \{0\}$ is anti-integral over R and $R[\alpha] = R[\mathbf{m}\alpha]$.*

PROOF. (a)$\Rightarrow$(b) follows from Corollary 1.1. (b)$\Rightarrow$(a):Any non-zero element α in K, $\alpha \in R$ or $1/\alpha \in R$ by Proposition 1, which asserts that R is a valuation domain. Since R is Noetherian, R is a DVR. □

LEMMA 2. *Let I be an ideal of R and P a prime ideal contaning I. Then if depth $R_P > 1$, then grade $IR_P > 1$.*

PROOF. We may assume that (R, **m**) is a local ring Take $a \in I$, $a \neq 0$ and let $aR = Q_1 \cap \ldots \cap Q_n$ be a primary decomposition with $P_i = \sqrt{Q_i}$. In this case depth $R_p = 1$ for all $P = P_i$. Since $P_i \not\supset I$ by assumption, we have $I \not\subset P_1 \cup \ldots \cup P_n$. Hence there exists $b \in I \setminus \cup P_i$. It is easy to see that $\{a,b\}$ is a regular sequence. Thus grade $I > 1$.

DEFINITION. *Let I be an ideal of R and let α be an element which belongs to some extension of R which is not necessarily equal to K. Consider the extension $A = R[I\alpha]$ of R. We say that A is a* ***pseudo-simple*** *extension of R defined by I and α if $I = R$ or depth $R_P > 1$ for any prime ideal P contaning I.*

Any simple extension $R[\alpha]$ (where α belongs to some extension of R) is a pseudo-simple extension of R.

In the next theorem, we use the following fact : for $\alpha \in K$, $R :_R \alpha$ is a divisorial ideal of R, that is, $R :_K (R :_R \alpha)$ contains α.

THEOREM 3. *Let α be an element in some extension of R. Assume that $A = R[I\alpha]$ is a pseudo-simple extension of R defined by I and α. If A is flat over R, then $A = R[\alpha]$, that is, A is a simple extension generated by α.*

PROOF. We may assume that (R, **m**) is a local ring. By Lemma 2, there exist $a, b \in I$ which forms a regular sequence. It is obvious that $a\alpha, b\alpha \in A$. Since A is flat over R, either a, b form an A-regular sequence or $(a, b)A = A$. In the former case, $A :_A \alpha$ contains a, b and hence $A :_A \alpha = A$ because $A :_A \alpha$ is a divisorial ideal of A. Thus $\alpha \in A$. In the later case, $(a,b)A = A$ implies that $1 = a\beta + b\gamma$ for some $\beta, \gamma \in A$. Thus $\alpha = (a\alpha)\beta + (b\alpha)\gamma \in A$. Therefore in any case, we conclude $\alpha \in A$. □

In the following examples, let k dente a field and let $k[x, y]$ denote a polynomial ring.

EXAMPLE 4.1. Let $R = k[x,y]$ and $\alpha = 1/xy$. Since $1/\alpha \in R$, $R = R(\alpha)$ and hence α is anti-integral over R. By definition, $R[I\alpha]$ is a pseudo-simple extension of R and $R[I\alpha] = R[(x,y)(1/xy)] = R[\alpha]$, a simple extension. Moreover since $1/\alpha \in R$, $R = R[1/\alpha] \rightarrow R[1/\alpha, \alpha] = R[\alpha]$ is obtained by localization and hence is flat. This shows that $R[I\alpha] = R[\alpha]$ does not always imply $I = R$.

EXAMPLE 4.2. Let R = k[x, y] and let A = R [x/y, y/x] = R [(x^2, y^2) (1/xy)], which is a pseudo-simple extension of R. It is not hard to see that A is not a simple extension of R. Since $1/\alpha = xy \in R$, α is anti-integral over R. By Theorem 3, A is not flat over R.

EXAMPLE 4.3. Let I be an ideal (x,y) of R = k[x,y] and let α is an element in some extension of R which satisfies $y^2\alpha^2+(x-1)\alpha+1 = 0$. Then $[K(\alpha):K] = 2$, that is, $\alpha \notin K$. R [Iα] is a pseudo-simple extension of R and R [Iα] = R[xα, yα]. Since $\alpha \in$ R [Iα], we have R [Iα] = R [α]. Since $1/\alpha$ is a zero of the single monic polynomial and $1/\alpha \in$ R [α], R $\to$ R [$1/\alpha$] is flat and R [$1/\alpha$] $\to$ R [$1/\alpha,\alpha$] = R [α] is a localization. Hence R $\to$ R [α] is flat.

LEMMA 5. *Let a be an element of R and let α be a non-zero element which is algebraic over R and let* $\phi_\alpha(X) = X^d+\mu_1X^{d-1}+\ldots+\mu_d$ *be the minimal polynomial of α in K[X]. Assume that the kernel of the canonical homomorphism R [X] → R [α] is generated by some polynomials of degree d. Put* $I_{[\alpha]} = \cap_{1\le i\le d}(R :_R \mu_i)$. *Then*
(i) *if* $a \in I_{[\alpha]}$, *then aα is integral over R;*
(ii) *if aα is integral over R, then* $a \in \sqrt{I_{[\alpha]}}$.

PROOF. (i) Put $b_i = a\mu_i \in R$. Then $a\alpha^d+b_1\alpha^{d-1}+\ldots+b_d = 0$ is a relation of α over R. From this, we get $(a\alpha)^d+ab_1(a\alpha)^{d-1}+\ldots+a^db_d = 0$. So a$\alpha$ is integral over R.
(ii) Let$(a\alpha)^n+c_1(a\alpha)^{n-1}+\ldots+c_n = 0$ be a relation of a α over R and put $h(X) = a^nX^n +c_1a^{n-1}+\ldots+c_n$. Then $h(\alpha) = 0$ and we can write $h(X) = \sum f_i(X)\, g_i(X)$ with $f_i(\alpha) = 0$, $\deg f_i(X) = d$ for all i. Since every coefficients of $f_i(X)$ belong to $I_{[\alpha]}$, $a^n \in I_{[\alpha]}$. Thus a $\in \sqrt{I_{[\alpha]}}$. □

PROPOSTITION 6. *Let α be a non-zero element which is algegraic over R and let A = R [Iα] be a pseudo-simple extension of R defined by I and α. Assume that the kernel of the canonical homomorphism R [X] → R [α] is generated by some polynomials of degree d. If A is integral over R, then α is intergral over R.*

PROOF. We have only to show that α is integral over R_P for all P $\in$ Spec(R). If P does not contain I, $A_P = R_P[\alpha]$ implies that α is integral over R_P. So we may assume that R is a local domain and I $\neq$ R. Since A is a pseudo-simple extension, we have grade I > 1, and hence there exists a regular sequence a, b in I. By assumption aα, bα are integral over R. Thus a, b $\in \sqrt{I_{[\alpha]}}$ by Lemma 5. But $\sqrt{I_{[\alpha]}}$ is a divisorial ideal, which implies that $\sqrt{I_{[\alpha]}}$ = R and hence $I_{[\alpha]}$ = R. Thus $\phi_\alpha(X) \in$ R [X]. Therefore α is integral over R.□

PROPOSITION 7. *Let A = R [Iα] be a pseudo-simple extension of R defined by I and α ∈ K. Assume that α is anti-integral over R. Then the following statements are equivalent:*

(a) *A is integral over R;*
(b) *α is integral over R;*
(c) $A = R$;
(d) $R = R[\alpha]$;
(e) $I \subset R :_R \alpha$.

PROOF. (a) ⇒ (b) follows from Proposition 6 . (b) ⇒ (d) follows from [3, (1. 3)]. (d) ⇒ (c), (d) ⇒ (b), (e) ⇒ (c) and (b) ⇒ (a) are trivial. □

COROLLARY 7.1. *Any proper pseudo-simple extension $R[I\alpha]$ of R defined by an ideal I and an anti-integral element α is not integral over R.*

REFERENCES

1) H. Matsumura, 'Commutative Algebra', Benjamin, New York (1970).
2) H. Matsumura, 'Commutative Ring Theory', Cambridge Univ. Press, Cambridge (1986).
3) S. Oda and K. Yoshida, Anti-integral extensions of Noetherian domains, Kobe J. Math, 5 (1988) 43—56.