Examples of Isospectral but not Isometric 24-dimensional Flat Tori.

Michihiko HASHIZUME
Department of Applied Mathematics, Faculty of Science
Okayama University of Science
Ridai-cho 1-1, Okayama 700, Japan
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0. Introduction.

The aim of this paper is to provide five pairs of isospectral but not isometric 24-dimensional flat tori. Our approach is quite similar to that of J. Milnor, who first constructed an example of isospectral but not isometric Riemannian manifolds using 16-dimensional flat tori (see[1]).

1. Generalities.

Let R^n be the n-dimensional Euclidean space. The canonical inner product (= the standard Riemannian structure) is denoted by $\langle\ ,\ \rangle$. An n-dimensional flat torus is a quotient Riemannian manifold $T(L) = R^n/L$ where L is a lattice in R^n.

1.1 Proposition[1]. *Two* n-*dimensional flat tori* $T(L_1)$ *and* $T(L_2)$ *are isometric iff there exists an isometry* f *of* R^n *satisfying* $f(L_1) = L_2$.

1.2 Definition. *Two lattices* L_1 *and* L_2 *in* R^n *are isomorphic iff there exists a* Z-*module isomorphism* f *of* L_1 *onto* L_2 *such that* $\langle f(x), f(y)\rangle = \langle x, y\rangle$ *for* $x, y \in L_1$.

The following corollary is a direct consequence of 1.1 and 1.2.

1.3 Corollary. *Two* n-*dimensional flat tori* $T(L_1)$ *and* $T(L_2)$ are isometric iff L_1 *and* L_2 *are isomorphic.*

The Laplace-Beltrami operator $\Delta(L)$ of T(L) is the one canonically induced from the Laplace-Beltrami operator $\Delta(R^n) = -\sum_{i=1}^{n}\partial^2/\partial x_i^2$ of R^n. We write Spec T(L) for the set of eigenvalues of $\Delta(L)$ counting with multiplicity. Note that Spec $T(L) = \{4\pi^2 < \xi, \xi > ; \xi \in L^*\}$ where L^* is the dual lattice of L.

1.4 Definition *Two* n-*dimensional flat tori* $T(L_1)$ *and* $T(L_2)$ *are called isospectral iff* Spec $T(L_1)$ = Spec $T(L_2)$.

For a lattice L in R^n we define the theta function $\Theta_L(z)$ by $\Theta_L(z) = \sum_{x\in L} e^{\pi i z<x,x>}$ where z is in the upper half plane. Since Spec $T(L) = \{4\pi^2 < \xi, \xi > ; \xi \in L^*\}$, we can conclude,

1.5 Proposition. *Two* n-*dimensional flat tori* $T(L_1)$ *and* $T(L_2)$ *are isospectral iff* $\Theta_{L_1^*}(z) = \Theta_{L_2^*}(z)$.

2. 24-dimensional flat tori.

We specify our consideration to flat tori defined by even integral unimodular lattices.

2.1 Definition. *A lattice* L *in* R^n *is called an even integral unimodular lattice* (EU *lattice in short*) *iff* $\langle x, y\rangle \in Z$ *for* $x, y \in L$, $\langle x, x\rangle \in 2Z$ *for* $x \in L$ *and* $L = L^*$.

Let L be an EU lattice in R^n. Note that n must be a multiple of 8[3]. Put $L(2k) = \{x \in L; \langle x, x\rangle = 2k\}$ for $k \in Z_+$. Then we have

$$\Theta_{L^*}(z) = \Theta_L(z) = 1 + \sum_{k=1}^{\infty} |L(2k)| e^{2\pi ikz}$$

where $|L(2k)|$ is the cardinality of $L(2k)$.

2.2 Proposition[3]. *Let* L *be an* EU *lattice in* R^{24}. *Then*

(1) $\Theta_L(z)$ *is a modular form of weight* 12 *for* SL(2, Z).

(2) $\Theta_L(z)$ *can be written as*

$$\Theta_L(z) = E_6(z) + (|L(2)| - 65520/691)\,\Delta(z)$$

where

$$E_6(z) = 1 + (65520/691) \sum_{k=1}^{\infty} \Big(\sum_{d|k} d^{11}\Big) e^{2\pi ikz}$$

and

$$\Delta(z) = e^{2\pi iz} \prod_{k=1}^{\infty} (1 - e^{2\pi ikz})^{24}.$$

Combining 1.5 with 2.2, we have

2.3 Corollary. *Let* L_1 *and* L_2 *be* EU *lattices in* R^{24}. *Then* $T(L_1)$ *and* $T(L_2)$ *are isospectral iff* $|L_1(2)| = |L_2(2)|$.

According to 1.3 and 2.3, our purpose is reduced to finding nonisomorphic pairs $\{L_1, L_2\}$ of EU lattices in R^{24} satisfying $|L_1(2)| = |L_2(2)|$.

On the other hand the complete calssification of the isomorphism classes of 24-dimensional EU lattices is given in[2] (see also[4]).

2.4 Proposition[2][4]. *Let* L *be an* EU *lattice in* R^{24}. *Then*

(I) L(2) *is either empty or forms a root system of rank* 24.

(II) *If* L(2) *is a root system, then its irreducible components consists of the root systems of type* A_n, D_n, E_6, E_7 *and* E_8, *and they have the common Coxeter number, say* h, *such that* $|L(2)| = 24h$.

(III) *There are exactly* 24 *isomorphism classes of* EU *lattices in* R^{24}. *The complete list of the representatives is given as follows;*

$L(\emptyset)$, $L(24A_1)$, $L(12A_2)$, $L(8A_3)$, $L(6A_4)$, $L(4A_6)$, $L(3A_8)$, $L(2A_{12})$, $L(A_{24})$, $L(6D_4)$, $L(4D_6)$, $L(3D_8)$, $L(2D_{12})$, $L(D_{24})$, $L(4E_6)$, $L(3E_8)$, $L(4A_5 + D_4)$, $L(2A_7 + 2D_5)$, $L(2A_9 + D_6)$, $L(A_{15} + D_9)$, $L(E_6 + D_7 + A_{11})$, $L(E_7 + A_{17})$, $L(2E_7 + D_{10})$, $L(E_8 + D_{16})$.

Here L(R) *with a root system* R *means an* EU *lattice such that* $L(2) = R$.

Since the Coxeter numbers of the root systems A_n, D_n, E_6, E_7 and E_8 are given respectively by $n + 1$, $2n - 2$, 12, 18, and 30, we can pick up all the pairs having the common $|L(2)|$. The results are as follows;

(I) $\{L(6D_4), L(4A_5 + D_4)\}$

(II) $\{L(4D_6), L(2A_9 + D_6)\}$

(III) $\{L(4E_6), L(E_6 + D_7 + A_{11})\}$

(IV) $\{L(2E_7 + D_{10}), L(E_7 + A_{17})\}$

(V) $\{L(3E_8), L(E_8 + D_{16})\}$

Now we can state our main results.

Theorem. *The following five pairs of* 24-*dimensional flat tori provide the examples of isospectral but not isometric flat tori*;

(I) $\{T(L(6D_4)), T(L(4A_5 + D_4))\}$

(II) $\{T(L(4D_6)), T(L(2A_9 + D_6))\}$

(III) $\{T(L(4E_6)), T(L(E_6 + D_7 + A_{11}))\}$

(IV) $\{T(L(2E_7 + D_{10})), T(L(E_7 + A_{17}))\}$

(V) $\{T(L(3E_8)), T(L(E_8 + D_{16}))\}$

References

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