Stability Property and Separation Condition in an Almost Periodic Integrodifferential Equation

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In order to discuss the existence of an almost periodic solution in an integrodifferential equation with infinite delay, we have discussed the relationship between the total stability with respect to a certain metric ρ and the separation condition with respect to ρ 2).

In this paper, we shall discuss a relationship between the ρ-separation condition and uniformly asymptotic stability property in a certain sense.

We shall consider a system of integrodifferential equations

$$\dot{x}(t)=f(t,\ x(t))+\int_{-\infty}^{0}F(t,\ s,\ x(t+s),\ x(t))ds, \tag{1}$$

where $f: R\times R^n \to R^n$ is continuous and is almost periodic in t uniformly for $x\in R^n$, and $F(t, s, x, y)$ is continuous on $R\times(-\infty,\ 0]\times R^n\times R^n$ and is almost periodic in t uniformly for $(s, x, y)\in R^*=(-\infty,\ 0]\times R^n\times R^n$. For the definition and the properties of almost periodic functions with parameters, see 4). If x is a function defined on $(-\infty, a)$, x_t is defined by the relation $x_t(s)=x(t+s)$, $-\infty<s\leqq 0$. Let $|x|$ be any norm of x in R^n. BC denotes the vector space of bounded continuous functions mapping $(-\infty,\ 0]$ into R^n, and for any ϕ, $\psi\in BC$, we set

$$\rho(\phi,\ \psi)=\sum_{m=1}^{\infty}\rho_m(\phi,\ \psi)/[2^m(1+\rho_m(\phi,\ \psi))],$$

where $\rho_m(\phi,\ \psi)=\sup_{-m\leqq s\leqq 0}|\phi(s)-\psi(s)|$. Clearly, $\rho(\phi^k,\ \phi)\to 0$ as $k\to\infty$ if and only if $\phi^k(s)\to\phi(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $k\to\infty$. Moreover, we denote by (BC, ρ) the space of bounded continuous functions $\phi: (-\infty,\ 0]\to R^n$ with metric ρ.

For system (1), we make the following assumptions :

(H_1) For any $\varepsilon>0$ and any compact set B in R^n, there exists an $S=S(\varepsilon,\ B)>0$

such that

$$\int_{-\infty}^{-S} |F(t,\ s,\ x(t+s),\ x(t))|\,ds \leqq \varepsilon \text{ for all } t\in R,$$

whenever $x(\sigma)$ is continuous and $x(\sigma)\in B$ for all $\sigma\leqq t$.

(H_2) System (1) has a bounded solution $u(t)$ defined on $[0,\ \infty)$ which passes through $(0,\ \phi^0)$, $\phi^0\in BC$.

Remark 1. It follows from (H_1) that for any compact set B in R^n, there exists an $L(B)>0$ such that

$$\int_{-\infty}^{0} |F(t,\ s,\ x(t+s),\ x(t))|\,ds \leqq L(B) \text{ for all } t\in R,$$

whenever $x(\sigma)$ is continuous and $x(\sigma)\in B$ for all $\sigma\leqq t$. Moreover, $\int_{-\infty}^{0} F(t,\ s,\ x(t+s),\ x(t))ds$ is continuous in t, whenever $x(\sigma)$ is continuous and bounded for $\sigma\leqq t$.

Under assumption (H_1), for any $t_0\in R$ and any $\phi\in BC$, there exists a solution of (1) which passes through $(t_0,\ \phi)$. Moreover, a solution $x(t)$ can be continuable up to $t=\infty$ if it remains in a compact set in R^n.

Denote by $\Omega(f,\ F)$ the set of all limiting functions $(g,\ G)$ such that for some sequence $\{t_k\}$, $t_k\to\infty$ as $k\to\infty$, $f(t+t_k,\ x)\to g(t,\ x)$ uniformly on $R\times S$ for any compact subset S in R^n and $F(t+t_k,\ s,\ x,\ y)\to G(t,\ s,\ x,\ y)$ uniformly on $R\times S^*$ for any compact subset S^* in R^* as $k\to\infty$. Then a system

$$\dot{x}(t)=g(t,\ x(t))+\int_{-\infty}^{0} G(t,\ s,\ x(t+s),\ x(t))ds \tag{2}$$

is called a limiting equation of (1) when $(g,\ G)\in\Omega(f,\ F)$. Clearly, if $(g,\ G)\in \Omega(f,\ F)$, $g(t,\ x)$ is almost periodic in t uniformly for $x\in\mathrm{R}^n$ and $G(t,\ s,\ x,\ y)$ is almost periodic in t uniformly for $(s,\ x,\ y)\in R^*$.

Remark 2. When $F(t,\ s,\ x,\ y)$ satisfies condition (H_1), any $G\in\Omega(F)$ satisfies condition (H_1) for the same $S=S(\varepsilon,\ B)>0$ as for F, that is,

$$\int_{-\infty}^{-S} |G(t,\ s,\ x(t+s),\ x(t))|\,ds \leqq \varepsilon \text{ for all } t\in R,$$

whenever $x(\sigma)$ is continuous and $x(\sigma)\in B$ for all $\sigma\leqq t$.

Let K be a compact set in R^n such that $u(t)\in K$ for all $t\in R$, where $u(t)=\phi^0(t)$ for $t<0$. If $x(t)$ is a solution such that $x(t)\in K$ for all $t\in R$, we say that x is in K.

Definition 1. We say that system (1) satisfies the ρ-separation condition in K, if for each $(g, G)\in\Omega(f, F)$, there exists a $\lambda(g, G)>0$ such that if x and y are distinct solutions of (2) in K, then we have

$$\rho(x_t, y_t)\geqq\lambda(g, G) \text{ for all } t\in R. \tag{3}$$

If system (1) satisfies the ρ-separation condition in K, then we can choose a positive constant λ_0 independent of (g, G) for which $\rho(x_t, y_t)\geqq\lambda_0$ for all $t\in R$, where x and y are distinct solutions of (2) in K. We shall call λ_0 the ρ-separation constant in K.

Definition 2. A solution $x(t)$ of (1) in K is said to be relatively totally (K, ρ)-stable, if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $\rho(x_t, y_t)<\varepsilon$ for all $t\geqq t_0$ whenever $\rho(x_{t_0}, y_{t_0})<\delta(\varepsilon)$ at some $t_0\in R$ and $p(t)$ is any continuous function which satisfies $|p(t)|<\delta(\varepsilon)$ for $t\geqq t_0$. Here y is a solution through (t_0, y_{t_0}) of

$$\dot{x}(t)=f(t, x(t))+\int_{-\infty}^{0}F(t, s, x(t+s), x(t))ds+p(t)$$

such that $y_{t_0}(s)\in K$ for $s\leqq 0$ and $y(t)\in K$ for $t\geqq t_0$.

In the case where $p(t)\equiv 0$, this gives the definition of the relative uniform (K, ρ)-stability of $x(t)$.

Hamaya and Yoshizawa 2) have obtained the following result.

Proposition. Under assumptions (H_1) and (H_2), if system (1) satisfies the ρ-separation condition in K, then for any $(g, G)\in\Omega(f, F)$, any solution x of (2) in K is relatively totally (K, ρ)-stable. Moreover, we can choose the number $\delta(\cdot)$ in Definition 2 so that $\delta(\varepsilon)$ depends only on ε and is independent of (g, G) and solutions.

Theorem 1. Under assumptions (H_1) and (H_2), suppose that system (1) satisfies the ρ-separation condition in K. If $w(t)$ is a solution of (1) such that $w(t)\in K$ for all $t\in R$, then $w(t)$ is almost periodic.

Proof. By proposition, solution $w(t)$ of (1) is relatively totally (K, ρ)-stable, because $(f, F) \in \Omega(f, F)$. Then $w(t)$ is asymptotically almost periodic on $[0, \infty)$ by Theorem 1 in 1). Thus, it has the decomposition $w(t)=p(t)+q(t)$, where $p(t)$ is almost periodic in t, $q(t)$ is continuous and $q(t) \to 0$ as $t \to \infty$. Since $w(t) \in K$ for all $t \in R$, $p(t)$ is a solution of (1) in K. If $w(t_1) \neq p(t_1)$ at some t_1, we have two distinct solutions of (1) in K. Thus we have $\rho(w_t, p_t) \geqq \lambda_0 > 0$ for all $t \in R$, where λ_0 is the ρ-separation constant. However, $w(t)$-$p(t) \to 0$ as $t \to \infty$, and hence $\rho(w_t, p_t) \to 0$ as $t \to \infty$. This contradiction shows $w(t) \equiv p(t)$ for all $t \in R$.

Definition 3. A solution $x(t)$ of (1) in K is said to be relatively uniformly asymptotically (K, ρ)-stable, if it is relatively uniformly (K, ρ)-stable and if there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that if $\rho(x_{t_0}, y_{t_0}) < \delta_0$ at some $t_0 \in R$, then $\rho(x_t, y_t) < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$, where y is a solution through (t_0, y_{t_0}) of (1) such that $y_{t_0}(s)$ for $s \leqq 0$ and $y(t) \in K$ for all $t \geqq t_0$.

We shall show that the ρ-separation condition will be characterized in terms of relatively uniformly asymptotic (K, ρ)-stability of solutions in K of limiting equations. For ordinary differential equations, this kind of problems has been discussed by Nakajima 3).

Theorem 2. Under assumptions (H_1) and (H_2), system (1) satisfies the ρ-separation condition in K if and only if for any $(g, G) \in \Omega(f, F)$, any solution x of (2) in K is relatively uniformly asymptotically (K, ρ)-stable with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$.

Proof. We suppose that system (1) satisfies the ρ-separation condition in K. Then it follows from proposition 1 that for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $(g, G) \in \Omega(f, F)$ and any solution $x(t)$ of (2) in K, if $\rho(x_{t_0}, y_{t_0}) < \delta(\varepsilon)$ at some $t_0 \in R$, then $\rho(x_t, y_t) < \varepsilon$ for all $t \geqq t_0$, where $y(t)$ is a solution of (2) such that $y_{t_0}(s) \in K$ for $s \leqq 0$ and $y(t) \in K$ for $t \geqq t_0$. Now let δ_0 be a positive constant such that $\delta_0 < \delta(\lambda_0/2)$, where λ_0 is the ρ-separation constant. For this δ_0, we shall show that for any $\varepsilon > 0$, there exists a $T(\varepsilon) > 0$ such that for any $(g, G) \in \Omega(f, F)$ and any solution $x(t)$ of (2) in K, $\rho(x_t, y_t) < \varepsilon$ for all $t \geqq t_0 + T(\varepsilon)$ whenever $\rho(x_{t_0}, y_{t_0}) < \delta_0$ at some $t_0 \in R$, where $y(t)$ is a solution of (2) such that $y_{t_0}(s) \in K$ for all $s \leqq 0$ and $y(t) \in K$ for all $t \geqq t_0$.

Suppose not. Then there exists an ε, $0<\varepsilon<\delta_0/2$, and sequences $\{(g_k, G_k)\}$, $\{x^k\}$, $\{y^k\}$, $\{s_k\}$ and $\{t_k\}$ such that $(g_k, G_k)\in\Omega(f, F)$, $x^k(t)$ is a solution in K of

$$\dot{x}(t)=g_k(t, x(t))+\int_{-\infty}^{0}G_k(t, s, x(t+s), x(t))ds \tag{4}$$

and that $t_k\geqq s_k+k$,

$$\rho(x^k_{s_k}, y^k_{s_k})<\delta_0<\delta(\lambda_0/2) \tag{5}$$

and

$$\rho(x^k_{t_k}, y^k_{t_k})\geqq\varepsilon, \tag{6}$$

where $y^k(t)$ is a solution of (4) such that $y^k_{s_k}(s)\in K$ for all $s\leqq 0$ and $y^k(t)\in K$ for all $t\geqq s_k$. Since (5) implies $\rho(x^k_t, y^k_t)<\lambda_0/2$ for all $t\geqq s_k$, we have

$$\varepsilon\leqq\rho(x^k_{t_k}, y^k_{t_k})\leqq\lambda_0/2. \tag{7}$$

If we set $w^k(t)=x^k(t+t_k)$ and $z^k(t)=y^k(t+t_k)$, then $w^k(t)$ is a solution in K of

$$\dot{x}(t)=g_k(t+t_k, x(t))+\int_{-\infty}^{0}G_k(t+t_k, s, x(t+s), x(t))ds \tag{8}$$

and $z^k(t)$ is defined for $t\geqq -k$ and is a solution of (8) such that $z^k_{-k}(s)\in K$ for all $s\leqq 0$ and $z^k(t)\in K$ for all $t\geqq -k$. Since $(g_k(t+t_k, x), G_k(t+t_k, s, x, y))\in\Omega(f, F)$, taking a subsequence if necessary, we can assume that $w^k(t)\to w(t)$ uniformly on any compact interval in R, $z^k(t)\to z(t)$ uniformly on any compact interval in R, $g_k(t+t_k, x)\to h(t, x)$ uniformly on $R\times K$ and $G_k(t+t_k, s, x, y)\to H(t, s, x, y)$ uniformly on $R\times S^*\times K\times K$ for any compact set S^* in $(-\infty, 0]$ as $k\to\infty$, where $(h, H)\in\Omega(f, F)$. Then, by the same argument as in the proof of Lemma 2 in 2), $w(t)$ and $z(t)$ are solutions in K of

$$\dot{x}(t)=h(t, x(t))+\int_{-\infty}^{0}H(t, s, x(t+s), x(t))ds. \tag{9}$$

On the other hand, we have

$$\rho(w_0, z_0)=\lim_{k\to\infty}\rho(w^k_0, z^k_0)=\lim_{k\to\infty}\rho(x^k_{t_k}, y^k_{t_k}).$$

Thus, it follows from (7) that

$$\varepsilon\leqq\rho(w_0, z_0)\leqq\lambda_0/2. \tag{10}$$

Since $w(t)$ and $z(t)$ are distinct solutions of (9) in K, (10) contradicts the ρ-separation condition. This shows that for any $(g, G)\in\Omega(f, F)$, any solution x of (2) in K is relatively uniformly asymptotically (K, ρ)-stable with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$.

Now we assume that for any $(g, G)\in\Omega(f, F)$, any solution of (2) in K is relatively uniformly asymptotically (K, ρ)-stable with a common triple $(\delta_0, \rho(\cdot), T(\cdot))$. First of all, we shall see that any two distinct solutions $x(t)$ and $y(t)$ in K of a limiting equation of (1) satisfy

$$\varliminf_{t\to-\infty} \rho(x_t, y_t) \geqq \delta_0 . \tag{11}$$

Suppose not. Then for some $(g, G)\in\Omega(f, F)$, there exist two distinct solutions $x(t)$ and $y(t)$ of (2) in K which satisfy

$$\varliminf_{t\to-\infty} \rho(x_t, y_t) < \delta_0 . \tag{12}$$

Since $x(t)\not\equiv y(t)$, we have $|x(t_0)-y(t_0)|=\varepsilon>0$ at some t_0. Thus we have $\rho(x_{t_0}, y_{t_0})\geqq\varepsilon/2(1+\varepsilon)$. By (12), there exists a t_1 such that $\rho(x_{t_1}, y_{t_1})<\delta_0$ and $t_1<t_0-T(\varepsilon/4(1+\varepsilon))$, where $T(\cdot)$ is the number for relatively uniformly asymptotic (K, ρ)-stability. Since $x(t)$ is relatively uniformly asymptotically (K, ρ)-stable, we have $\rho(x_{t_0}, y_{t_0})<\varepsilon/4(1+\varepsilon)$, which contradicts $\rho(x_{t_0}, y_{t_0})\geqq\varepsilon/2(1+\varepsilon)$. Thus we have (11).

For any solution $x(t)$ in K, there exist positive constants c and L^* such that $|x(t)|\leqq c$ and $|\dot{x}(t)|\leqq L^*$ for all $t\in R$. Denote by X the set

$$X=\{\phi\in BC : \phi(s) \text{ is a function such that } |\phi(s)|\leqq c \text{ for } s\in(-\infty, 0] \text{ and } |\phi(s_1)-\phi(s_2)|\leqq L^*|s_1-s_2| \text{ for all } s_1, s_2\in(-\infty, 0]\}.$$

Then X is compact in (BC, ρ). Thus, there are finite number of coverings which consist of m_0 balls with diameter $\delta_0/4$. We shall see that the number of distinct solutions of (2) in K is at most m_0. Suppose that there are m_0+1 distinct solutions $x^{(j)}(t)$ $(j=1, 2, \cdots, m_0+1)$. By (11), there exists a t_2 such that

$$\rho(x_{t_2}^{(i)}, x_{t_2}^{(j)})\geqq\delta_0/2 \text{ for } i\neq j. \tag{13}$$

Since $x_{t_2}^{(j)}$, $j=1, 2, \cdots, m_0+1$ are in X, some two of these, say $x_{t_2}^{(i)}$, $x_{t_2}^{(j)}$, $(i\neq j)$, are in one ball and hence $\rho(x_{t_2}^{(i)}, x_{t_2}^{(j)})<\delta_0/4$, which contradicts (13). Therefore the number of solutions of (2) in K is $m\leqq m_0$. Thus we have the set of solutions of (2) in K

$$\{x^{(1)}(t), x^{(2)}(t), \cdots, x^{(m)}(t)\}$$

and

$$\varliminf_{t\to-\infty} \rho(x_t^{(i)}, x_t^{(j)})\geqq\delta_0 \text{ for } i\neq j. \tag{14}$$

Consider a sequence $\{t_k\}$ such that $t_k\to-\infty$, $g(t+t_k, x)\to g(t, x)$ uniformly on

$R \times K$ and $G(t+t_k, s, x, y) \to G(t, s, x, y)$ uniformly on $R \times S^* \times K \times K$ for any compact set S^* in $(-\infty, 0]$ as $k \to \infty$. Since the sequences $\{x^{(j)}(t+t_k)\}$, $1 \leqq j \leqq m$, are uniformly bounded and equicontinuous, there exists a subsequence of $\{t_k\}$, which will be denoted by $\{t_k\}$ again, and functions $y^{(j)}(t)$ such that $x^{(j)}(t+t_k) \to y^{(j)}(t)$ uniformly on any compact interval in R as $k \to \infty$. Clearly $y^{(j)}(t)$ is solution of (2) in K. Since we have

$$\rho(y_t^{(i)}, y_t^{(j)}) = \lim_{k\to\infty} \rho(x_{t+t_k}^{(i)}, x_{t+t_k}^{(j)}) \text{ for } t \in R,$$

it follows from (14) that

$$\rho(y_t^{(i)}, y_t^{(j)}) \geqq \delta_0 \text{ for all } t \in R \text{ and } i \neq j. \tag{15}$$

Since we have (15), distinct solutions of (2) in K are $y^{(1)}(t)$, $y^{(2)}(t)$, $\cdots$, $y^{(m)}(t)$. This shows that system (1) satisfies the ρ-separation condition in K with the ρ-separation constant δ_0.

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