On Valuation Ideals and Integrally Closed Ideals

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§ 0. INTRODUCTION

Let R be an integral domain with quotient field K and let I be an ideal of R. The concept of integrally closedness was introduced by Zariski- Samuel and Nagata (cf. [3], [5]). In particular, Zariski and Samuel (cf. [5, Appendix 4]) proved that I is an integrally closed ideal if and only if I is a valuation ideal.

In this paper, we shall try to simplify this result using the theory of Rees rings. Further, for this application, we shall prove some results concerning the intersection and the product of ideals.

Throughout this paper all rings will be commutative integral domain with identities and ideals assumed to be finitely generated.

§ 1. VALUATION IDEAL AND INTEGRALLY CLOSED IDEAL

At first, we give the definition of valuation ideal.

Definition 1.1. Let $\mathfrak{a}$ be an ideal of R and let $V_1, \cdots, V_n$ be some valuation rings containing R. If there exists a primary ideal Q_i of V_i and

$$\mathfrak{a} = Q_1 \cap \cdots \cap Q_n \cap R,$$

then we call that the ideal $\mathfrak{a}$ is a valuation ideal.

Remark 1.2. (1) The quotient field of V_i is not necessarily equal to the quotient field of R. But we may assume that the two fields are the same.

Indeed, let $K = \boldsymbol{Q}(R)$ be the quotient field and $W_i = V_i \cap K$, $Q'_i \cap W_i$. *Then*

$$\begin{aligned} & Q'_1 \cap \cdots \cap Q'_n \cap R \\ &= (Q_1 \cap W_1) \cap \cdots \cap (Q_n \cap W_n) \cap R \\ &= (Q_1 \cap V_1 \cap K) \cap \cdots \cap (Q_n \cap V_n \cap K) \cap R \end{aligned}$$

$=Q_1\cap\cdots\cap Q_n\cap V_1\cap\cdots\cap V_n\cap K\cap R.$

Since $V_1\cap\cdots\cap V_n\cap K\cap R=R$, *we have* $Q'_1\cap\cdots\cap Q'_n\cap R=\mathfrak{a}$.

(2) *We may show that*

$$\mathfrak{a}=\mathfrak{a}V_1\cap\cdots\cap\mathfrak{a}V_n\cap R.$$

Indeed, it is easy to see that

$$\mathfrak{a}\subseteqq\mathfrak{a}V_1\cap\cdots\cap\mathfrak{a}V_n\cap R.$$

Since $\mathfrak{a}=Q_1\cap\cdots\cap Q_n\cap R$, *we have* $\mathfrak{a}\subseteqq Q_i$, *thus*

$$\mathfrak{a}V_1\cap\cdots\cap\mathfrak{a}V_n\cap R\subseteqq Q_1\cap\cdots\cap Q_n\cap R=\mathfrak{a}.$$

Next, we define an integrally closed ideal. After we can show that the valuation ideal is equivalent to the integrally closed ideal.

Definition 1.3. For an ideal $\mathfrak{a}$ *of* R, *we define that*

$$\overline{\mathfrak{a}}=\{\alpha\in R \mid n:\text{ integer},\ a_i\in\mathfrak{a}^i,\ 1\leqq i\leqq n,$$
$$\alpha^n+a_1\alpha^{n-1}+\cdots+a_n=0\},$$

and $\overline{\mathfrak{a}}$ *is called the integral closure of* $\mathfrak{a}$.

If $\overline{\mathfrak{a}}=\mathfrak{a}$, *then* $\mathfrak{a}$ *is called the integrally closed ideal.*

When R is integrally closed as the ring, the principal ideal aR is integrally closed as the ideal.

Now, we can state and prove some results for preparation.

Lemma 1.4. If two ideals $\mathfrak{a}$ *and* $\mathfrak{L}$ *are integrally closed, then* $\mathfrak{a}\cap\mathfrak{L}$ *is also.*

Proof. By definition, we have $\mathfrak{a}\cap\mathfrak{L}\subset\mathfrak{a}$. Therefore, it is easy to see that $\overline{\mathfrak{a}\cap\mathfrak{L}}\subset\overline{\mathfrak{a}}$.. Hence

$$\overline{\mathfrak{a}\cap\mathfrak{L}}\subseteqq\overline{\mathfrak{a}}\cap\overline{\mathfrak{L}}\subseteqq\mathfrak{a}\cap\mathfrak{L}=\mathfrak{a}\cap\mathfrak{L}.$$

Thus $\mathfrak{a}\cap\mathfrak{L}$ is integrally closed.

Lemma 1.5. Let A be a ring extension of R. If I is an integrally closed ideal of A, then $\mathfrak{a}=I\cap R$ *is also in R.*

Proof. Let α be an element of $\overline{\mathfrak{a}}$, there exists an integer n and $a_i \in \mathfrak{a}^i$ such that

$$\alpha^n + a_l \alpha^{n-1} + \cdots + a_n = 0.$$

Therefore $a_i \in \mathfrak{a}^i \subseteqq I^i$, $\alpha \in \overline{I} = I$. Hence $\alpha \in I \cap R = \mathfrak{a}$.

The following result is well-known (see [5]), but we give a simple proof for the completeness.

Theorem 1.6. Let $\mathfrak{a}$ *be an ideal of R. Then the followings are equivalent to each other;*

(1) $\mathfrak{a}$ *is a valuation ideal.*

(2) $\mathfrak{a}$ *is an integrally closed ideal.*

Proof. We assume that $\mathfrak{a}$ is a valuation ideal. By Remark 1.2, there exist some valuation rings $V_1, \cdots, V_n$ such that

$$\begin{aligned}\mathfrak{a} &= \mathfrak{a} V_1 \cap \cdots \cap \mathfrak{a} V_n \cap R \\ &= (\mathfrak{a} V_1 \cap R) \cap \cdots \cap (\mathfrak{a} V_n \cap R).\end{aligned}$$

Let $V = V_i$ be a valuation ring. By Lemma 1.4, it suffices to show that $\mathfrak{a} V \cap R$ is integrally closed.

Let α be an element of $\overline{\mathfrak{a}}$, there exists an integer n and $a_i \in \mathfrak{a}^i$ such that

$$\alpha^n + a_1 \alpha^{n-1} + \cdots + a_n = 0.$$

Since R is noetherian, $\mathfrak{a}$ is finitely generated and V is a valuation ring, so $\mathfrak{a} V = bV$ for some $b \in \mathfrak{a}$. Therefore $a_i \in \mathfrak{a}^i V = b^i V$, thus $a_i = b^i \beta_i$ for some $\beta_i \in V$. Hence

$$\alpha^n + \beta_1 b \alpha^{n-1} + \cdots + \beta_n b^n = 0.$$

We have a monic relation of the form:

$$\left(\frac{\alpha}{b}\right)^n + \beta_1 \left(\frac{\alpha}{b}\right)^{n-1} + \cdots + \beta_n = 0$$

Since V is integrally closed, $\frac{\alpha}{b} \in V$. Therefore

Hence $\mathfrak{a}$ is integrally closed.

To show the converse implication of Theorem 1.6, we give the following definition;

Definition 1.7. Let t be an indeterminant and let $A_{\mathfrak{a}}=R[t^{-1}, \mathfrak{a}t]$ be a Rees algebra over R. Since $A_{\mathfrak{a}}$ is finitely generated over R, $A_{\mathfrak{a}}$ is noetherian. Hence the integral closure $\overline{A}_{\mathfrak{a}}$ is a Krull domain.

For a convenience, let $u=t^{-1}$ and $\mathfrak{p}_1, \cdots, \mathfrak{p}_k$ be the prime divisors of $u\overline{A}_{\mathfrak{a}}$ and put $V_i=(\overline{A}_{\mathfrak{a}})_{\mathfrak{p}_i}$. Then V_i is a discrete valuation ring.

We shall show that $\overline{\mathfrak{a}}=u\overline{A}_{\mathfrak{a}}\cap R$.
Let a be an element of $\overline{\mathfrak{a}}$. Then we have a monic relation of the form:

$$a^n+a_1a^{n-1}+\cdots+a_n=0,$$

for some $a_i\in\mathfrak{a}^i$. By multiplying t^n,

$$(at)^n+a_1t\ (at)^{n-1}+\cdots+a_n t^n=0.$$

Since $a_i t^i\in A_{\mathfrak{a}}$, $at\in\overline{A}_{\mathfrak{a}}$, *thus we have* $a\in u\overline{A}_{\mathfrak{a}}\cap R$.

Conversely, let $a\in u\overline{A}_{\mathfrak{a}}\cap R$, *then* $at\in\overline{A}_{\mathfrak{a}}$. *Hence there exists a monic relation*

$$(at)^n+b_1(at)^{n-1}+\cdots+b_n=0,$$

where b_i are elements of $A_{\mathfrak{a}}$. Since the Rees algebra is a graded ring, we have only to consider the degree n part. Hence we may assume that $b_i=a_it_i$ $(1\leqq i\leqq n$, $a_i\in\mathfrak{a}^i)$. *It holds that*

$$(at)^n+a_lt(at)^{n-1}+\cdots+a_nt^n=0.$$

For $a_i\in\mathfrak{a}^i$, we have $a\in\overline{\mathfrak{a}}$.

The ring $\overline{A}_{\mathfrak{a}}$ is a Krull domain, so we have the primary decomposition

$$u\overline{A}_{\mathfrak{a}}=u\ (\overline{A}_{\mathfrak{a}})_{\mathfrak{p}_l}\cap\cdots\cap u\ (\overline{A}_{\mathfrak{a}})_{\mathfrak{p}_k}\cap\overline{A}_{\mathfrak{a}}$$
$$=uV_1\cap\cdots\cap uV_k\cap\overline{A}_{\mathfrak{a}}.$$

Since $u\overline{A}_{\mathfrak{a}}\cap R=\overline{\mathfrak{a}}$, we have $\overline{\mathfrak{a}}=uV_1\cap\cdots\cap uV_k\cap R$.
The integrally closedness of $\mathfrak{a}$ implies $\mathfrak{a}=\overline{\mathfrak{a}}$, that is, $\mathfrak{a}$ is a valuation ideal.

Remark 1.8. By the above proof, it is seen that the valuation ring may be a discrete valuation ring. If $\mathfrak{a}$ is integrally closed, we have

$$\mathfrak{a}=\mathfrak{a}V_1\cap\cdots\cap\mathfrak{a}V_k\cap R$$

Lemma 2.3. Let $\mathfrak{a}$ *be an ideal of R and let* $\mathfrak{p}_1 \cdots, \mathfrak{p}_t$ *be the prime divisors of* $\mathfrak{a}$. *The following conditions are equivalent;*

(1) $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$.

(2) $\mathfrak{L} \not\subset \mathfrak{p}_i$ *for all* i.

Proof. Suppose that $\mathfrak{L} \subset \mathfrak{p}_1$. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be an irredundant primary decomposition of $\mathfrak{a}$. Then $\mathfrak{a} : \mathfrak{L} \supseteqq \mathfrak{a} : \mathfrak{p}_1$ holds. We shall show that $\mathfrak{a} : \mathfrak{p}_1 \supsetneqq \mathfrak{a}$. Since $\mathfrak{q}_1 \not\supset \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$, there exists an element x such that $x \notin \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$. Hence $x\ \mathfrak{p}_1^k \subseteqq \mathfrak{q}_1$ and $x \mathfrak{p}_1^{k-1} \not\subset \mathfrak{q}_1$ for some integer k. Therefore there exists an element y such that $y \in x \mathfrak{p}_1^{k-1}$ and $y \notin \mathfrak{q}_1$. Then $\mathfrak{p}_1\ y \subseteqq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t = \mathfrak{a}$ and $y \in \mathfrak{a}$ for $y \notin \mathfrak{q}_1$. Therefore we have $\mathfrak{a} : \mathfrak{p}_1 \supsetneqq \mathfrak{a}$. Hence $\mathfrak{a} : \mathfrak{L} \supsetneqq \mathfrak{a}$.

Next, we assume that $\mathfrak{L} \not\subset \mathfrak{p}_i$ for all i. Let x be an element of $\mathfrak{a} : \mathfrak{L}$. There exists some element $b_i \in \mathfrak{L}$ not contained in $\mathfrak{p}_i$. Then $xb_i \notin \mathfrak{p}_i$, we have $x \in \mathfrak{q}_i$ for all i. Hence $x \in \mathfrak{a}$. Therefore $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$.

Remark 2.4. If $\mathfrak{a} : \mathfrak{L}$, then the prime divisors $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$ of $\mathfrak{a}$ are also of $\mathfrak{a}\mathfrak{L}$.

Indeed, we may consider the localization of R. Since $\mathfrak{L} \not\subset \mathfrak{p}_i$,

$\mathfrak{L}_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ and $(\mathfrak{a}\mathfrak{L})_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i} \mathfrak{L}_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i}$.

Hence $\mathrm{Ass}_R\ (R/\mathfrak{a}) \subseteqq \mathrm{Ass}_R\ (R/\mathfrak{a}\mathfrak{L})$.

Proposition 2.5. Assume that $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$ *and* $\mathfrak{L} : \mathfrak{a} = \mathfrak{L}$. *Moreover if*

$$Ass_R\ (R/\mathfrak{a}\mathfrak{L}) = Ass_R\ (R/\mathfrak{a}) \cup Ass_R\ (R/\mathfrak{L}),$$

then $\mathfrak{a} \cap \mathfrak{L} = \mathfrak{a}\mathfrak{L}$ *holds.*

Proof. It is easily seen that $\mathfrak{a} \cap \mathfrak{L} \supseteqq \mathfrak{a}\mathfrak{L}$. Let $Ass_R\ (R/\mathfrak{a}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_t\}$ and $Ass_R\ (R/\mathfrak{L}) = \{\mathfrak{p}'_1, \cdots\cdots, \mathfrak{p}'_s\}$. By the assumption, $Ass_R\ (R/\mathfrak{a}\mathfrak{L}) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_t, \cdots, \mathfrak{p}'_s\}$. Therefore we have a primary decomposition

$$\mathfrak{a}\mathfrak{L} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t \cap \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_s,$$

where $\mathfrak{q}_1$ (resp. $\mathfrak{q}'_j$) is a primary ideal belonging to $\mathfrak{p}_i$ (resp. $\mathfrak{p}'_j$). We shall show that $\mathfrak{a} \cap \mathfrak{L}$ is contained in $\mathfrak{q}_i$ and $\mathfrak{q}'_j$. The assumption $\mathfrak{a}$ implies $\mathfrak{L} \not\subset \mathfrak{p}_i$ by the Lemma 2.3. Hence

$$(\mathfrak{a} \cap \mathfrak{L})_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i} \cap \mathfrak{L}_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i}.$$

Therefore $\mathfrak{a} \cap \mathfrak{L} \subseteqq (\mathfrak{a} \cap \mathfrak{L})_{\mathfrak{p}_i} = \mathfrak{a}_{\mathfrak{p}_i} \subseteqq \mathfrak{q}_i R_{\mathfrak{p}_i}$.

by Remark 1.2. Since ideals of the valuation ring V are all primary ideals, of the valuation ring V are all primary ideals, each ideals Q_i is a primary ideal of V_i. Thus $Q_i \cap R$ is also in R. Therefore

$$\begin{aligned}\mathfrak{a} &= Q_1 \cap \cdots \cap Q_n \cap R \\ &= (Q_1 \cap R) \cap \cdots \cap (Q_n \cap R)\end{aligned}$$

is a primary decomposition of $\mathfrak{a}$ and $Q_i \cap R$ is an integrally closed ideal of R.

Theorem 1.9. An integrally closed ideal has a decomposition consists of integrally closed prilary ideals.

§2. INTERSECTION AND PRODUCT OF IDEALS

The following proposition is well-known.

Proposition 2.1. If two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, i.e., $\mathfrak{a} + \mathfrak{b} = R$, then $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

The condition $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ not necessarily implies that $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal, but holds. When $\mathfrak{b}$ is a principal ideal, we see that;

Proposition 2.2. Let R be an integral domain and $\mathfrak{b} = bR$ $(b \in R)$. then followings are equivalent to each other;

(1) $\mathfrak{a} : b = \mathfrak{a}$.

(2) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

Proof. Assume that $\mathfrak{a} : b = \mathfrak{a}$. It is easily seen that $\mathfrak{a} \cap \mathfrak{b} \supseteqq \mathfrak{a}\mathfrak{b}$. Let y be an element of $\mathfrak{a} \cap \mathfrak{b}$. As $y \in \mathfrak{b}$, there exists an element x such that $y = bx$. Since y is in $\mathfrak{a}$, $xb \in \mathfrak{a}$. Therefore $x \in \mathfrak{a} : b = \mathfrak{a}$. Hence $bx \in \mathfrak{a}\mathfrak{b}$,

Conversely, we assume that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$. It is easily to see that $\mathfrak{a} : b \supseteqq \mathfrak{a}$. Let x be an element of $\mathfrak{a} : b$. Then.

$$xb \in \mathfrak{a} \cap bR = \mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b} = b\mathfrak{a}.$$

Therefore $xb \in b\mathfrak{a}$, that is, $x \in \mathfrak{a}$. Hence we have $\mathfrak{a} : b = \mathfrak{a}$.

We consider the condition of $\mathfrak{a} : \mathfrak{b} = \mathfrak{a}$.

Hence $\mathfrak{a} \cap \mathfrak{L} \subseteqq \mathfrak{q}_i$.
By the same reason, we have $\mathfrak{a} \cap \mathfrak{L} \subseteqq \mathfrak{q}'_j$.

Proposition 2.6. Let $\mathfrak{a}$ be an integrally closed ideal and let $\mathfrak{L}$ be an ideal of R. If $\mathfrak{a} \cap \mathfrak{L} = \mathfrak{a}\mathfrak{L}$, then $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$.

Proof. It is easily seen that $\mathfrak{a} : \mathfrak{L} \supseteqq \mathfrak{a}$. Since $\mathfrak{a}$ is an integrally closed ideal, $\mathfrak{a}$ is a valuation ideal by Theorem 1.6. Therefore there exist some discrete valuations $v_1, v_2, \cdots, v_t$ and some positive integers $e_1, e_2, \cdots, e_t$ such that $x \in \mathfrak{a}$ if and only if $v_i(x) \geqq e_i$ for all i.
Let x be an element of $\mathfrak{a} : \mathfrak{L}$. Then $x\mathfrak{L} \subseteqq \mathfrak{a} \cap \mathfrak{L} = \mathfrak{a}\mathfrak{L}$. For any i,

$$\begin{aligned} v_i(x\mathfrak{L}) &= v_i(x) + v_i(\mathfrak{L}) \\ &= v_i(\mathfrak{a}\mathfrak{L}) \\ &= v_i(\mathfrak{a}) + v_i(\mathfrak{L}) \\ &= e_i + v_i(\mathfrak{L}). \end{aligned}$$

Hence we have $v_i(x) \geqq e_i$, that is, $x \in \mathfrak{L}$. Therefore $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$.

Now, we obtain the following;

Theorem 2.7. Let $\mathfrak{a}$ and $\mathfrak{L}$ be the integrally closed ideals such that

$$Ass_R(R/\mathfrak{a}\mathfrak{L}) \subseteqq Ass_R(R/\mathfrak{a}) \cup Ass_R(R/\mathfrak{L}).$$

Then the folowings are equivalent;

(1) $\mathfrak{a} \cap \mathfrak{L} = \mathfrak{a}\mathfrak{L}$.

(2) $\mathfrak{a} : \mathfrak{L} = \mathfrak{a}$, $\mathfrak{L} : \mathfrak{a} = \mathfrak{L}$.

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