

An Octonionic Representation of $SO(6)$ Spinors

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An algebra of octonions to be covariant under the $Spin(6)$ -transformation is derived. It is shown that these octonions can be regarded as a representation of $SO(6)$ spinors.

1. Introduction

The first indication of the significance of octonion algebras to physics is perhaps due to Pais¹⁾. He showed that there are some remarkable similarities between the structure of interactions of elementary particles and the algebra of octonions. Gunaydin and Gursev²⁾ studied, using an octonion algebra, the quark structure. On the other hand, together with the development of the eleven-dimensional supergravity theory, octonion algebras on S^7 were studied in detail by many particle physicists³⁾. All these works are based on G_2 or $SO(7)$ symmetry of octonions.

Recent developments of the ten-dimensional physics caused by the superstring theory give us some important suggestions that the octonion algebra is a mathematical language to express the nature⁴⁾. Kugo and Townsend⁵⁾ suggested in the first place this fact, but they could not completely prove it because of the difficulty due to the nonassociativity of the octonions. After Kugo and Townsend, several authors studied the connection between the ten-dimension and the octonions⁶⁾. Tachibana and Imaeda⁷⁾ proved completely that components of spinors in the ten-dimensional Minkowski space-time are represented by the octonions on which the $Spin(8)$ -group acts.

To make the superstring theory realistic, ten-dimensional space-time should be reduced to an external four-dimensional space-time and an internal compacted six-dimensional space⁸⁾. Then an octonionic spinor in the ten-dimensional space-time should be decomposed into a spinor in the four-dimensional space-time and a spinor in the six-dimensional space. We deduce that the spinor in the six-dimensional space, one of the decomposed spinors, is represented by an octonion.

Moreover, this decomposition guarantees a justification of the octonionic representations of the ten-dimensional spinors.

In this paper, we describe an octonionic representation of the spinors in the six-dimensional space. Since the six-dimensional space of which we are interested is Euclidean, the group which acts on the vectors in this space is $SO(6)$. Therefore, we will describe the representation in terms of octonions of $SO(6)$ spinors.

2. Preliminaries

We express an octonion by

$$\mathbf{A} = a_0 + \mathbf{i}_1 a_1 + \dots + \mathbf{i}_7 a_7, \quad (1)$$

wher a_0, a_1, \dots, a_7 are real numbers and $\mathbf{i}_1, \dots, \mathbf{i}_7$ are the octonion imaginary units which satisfy the conditions

$$\begin{aligned} \mathbf{i}_b \mathbf{i}_c &= -\delta_{bc} + \mathbf{i}_a \varepsilon_{abc}, (a, b, c, \dots = 1, 2, \dots, 7), \\ \varepsilon_{abc} &= -\varepsilon_{bac} = \varepsilon_{bca}, \\ \varepsilon_{123} &= \varepsilon_{145} = \varepsilon_{176} = \varepsilon_{246} = \varepsilon_{257} = \varepsilon_{347} = \varepsilon_{365} = 1, \\ &\text{otherwise } \varepsilon_{abc} = 0 \end{aligned} \quad (2)$$

The octonion conjugation of \mathbf{A} is denoted by $\overline{\mathbf{A}}$, given by

$$\overline{\mathbf{A}} = a_0 - \mathbf{i}_1 a_1 - \dots - \mathbf{i}_7 a_7 \quad (3)$$

Then $\overline{\mathbf{AB}} = \overline{\mathbf{B}} \overline{\mathbf{A}}$ holds for two octonions \mathbf{A} and \mathbf{B} .

The octonion algebra is noncommutative and nonassociative; for three octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

$$\mathbf{AB} - \mathbf{BA} \neq 0, \quad (4)$$

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] := (\mathbf{AB})\mathbf{C} - \mathbf{A}(\mathbf{BC}) \neq 0, \quad (5)$$

where $[\mathbf{A}, \mathbf{B}, \mathbf{C}]$ is called the associator of octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$. We have for the associator the following formulas;

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = -[\mathbf{B}, \mathbf{A}, \mathbf{C}] = -[\mathbf{A}, \mathbf{C}, \mathbf{B}] = -[\overline{\mathbf{A}}, \mathbf{B}, \mathbf{C}] = -[\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\mathbf{C}}], \quad (6)$$

$$[\mathbf{A}, \mathbf{BA}, \mathbf{C}] = [\mathbf{A}, \overline{\mathbf{A}}\mathbf{B}, \mathbf{C}] = \mathbf{A}[\mathbf{A}, \mathbf{B}, \mathbf{C}] = [\mathbf{A}, \mathbf{B}, \mathbf{C}]\overline{\mathbf{A}}. \quad (7)$$

From the formulas (6) and (7), we obtain the following important identity (Moffang);

$$(\mathbf{AB})(\mathbf{CA}) = (\mathbf{A}(\mathbf{BC}))\mathbf{A} = \mathbf{A}((\mathbf{BC})\mathbf{A}) =: \mathbf{A}(\mathbf{BC})\mathbf{A}, \quad (8)$$

for three octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

We define the scalar product of two octonions \mathbf{A} and \mathbf{B} as follows;

$$\mathbf{A} \cdot \mathbf{B} := \frac{1}{2}(\mathbf{A}\overline{\mathbf{B}} + \mathbf{B}\overline{\mathbf{A}}) = \frac{1}{2}(\overline{\mathbf{A}}\mathbf{B} + \overline{\mathbf{B}}\mathbf{A}) = \mathbf{B} \cdot \mathbf{A}. \quad (9)$$

Then, for three octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$, we have the identities

$$\mathbf{A} \cdot (\mathbf{BC}) = \mathbf{B} \cdot (\mathbf{A}\bar{\mathbf{C}}) = \mathbf{C} \cdot (\bar{\mathbf{B}}\mathbf{A}). \quad (10)$$

Moreover, we define the norm of an octonion \mathbf{A} by $N(\mathbf{A})$;

$$N(\mathbf{A}): = \mathbf{A} \cdot \mathbf{A} = \mathbf{A}\bar{\mathbf{A}} = \bar{\mathbf{A}}\mathbf{A} = a_0^2 + a_1^2 + \dots + a_7^2. \quad (11)$$

Then we can prove, for two octonions \mathbf{A} and \mathbf{B} ,

$$N(\mathbf{AB}) = N(\mathbf{A})N(\mathbf{B}). \quad (12)$$

From (11) and (12), we see that the octonion algebra is one of the divisions algebras.

The formulas above described hold for any octonions. However, we need in this paper the other algebraic properties of octonions which are not “general (or full)” octonions but are “restricted” octonions. We describe here some properties for these octonions.

Proposition 1. *For octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$, an equation*

$$\mathbf{A}(\mathbf{BC}) = \mathbf{B}(\bar{\mathbf{A}}\mathbf{C}) \quad (13)$$

holds, if and only if \mathbf{A} consists of the real part and the n (where $n \leq 6$) imaginary parts and \mathbf{B} consists of only the imaginary parts which do not have the same imaginary units as \mathbf{A} .

Proposition 2. *For octonions $\mathbf{A}, \mathbf{B}, \mathbf{C}$, an equation*

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \quad (14)$$

holds, if and only if \mathbf{A} and \mathbf{B} consist of a real part and an imaginary part and have the common imaginary units.

These propositions can be proved by using the formulas to the associator.

3. The Six-octonions

Let V_i ($i, j, \dots = 1, 2, \dots, 6$) be a component of a vector with respect to the orthonormal basis in the six-dimensional Euclidean space \mathbf{R}^6 . Consider a hypercomplex number

$$\mathbf{V} = \mathbf{i}_i V_i, \quad (15)$$

called as the “six-octonion”, where \mathbf{i}_i are octonion imaginary units excluding \mathbf{i}_7 . Consider for a six-octonion \mathbf{V} the following octonionic transformation;

$$V \rightarrow i_i (\mathbf{w}_{(ij)} V \mathbf{w}_{(ij)}) i_i, \quad (\text{no sum over } i, j) \quad (16)$$

$$\mathbf{w}_{(ij)} = i_i \cos(\theta/2) + i_j \sin(\theta/2). \quad (17)$$

The transformation (16) induces for the coefficients with respect to the octonion imaginary units of V the following changes;

$$\begin{aligned} V_i &\rightarrow V_i \cos \theta - V_j \sin \theta, \\ V_j &\rightarrow V_i \sin \theta + V_j \cos \theta, \\ V_k &\rightarrow V_k, \quad (k \neq i, j). \end{aligned} \quad (18)$$

Thus the transformation (16) is equivalent to a rotation of (i, j) -plane through an angle θ in \mathbf{R}^6 . Taking another six-octonion U produced by the same procedure as V , we can easily prove that the scalar product of U and V is invariant under the transformation (16);

$$U \cdot V = U_i V_i = \text{inv.} \quad (19)$$

Note that a six-octonion transformed under the transformation (16) is also a six-octonion.

Proposition 3. If a six-octonion V transforms under the transformation (16), then V is the octonionic representation of a six-dimensional vector on which the $SO(6)$ group acts.

4. The Spin(6)-octonions

Let us consider a "full" octonion Ψ which transforms under the transformation (16) as follows:

$$\Psi \rightarrow i_i (\mathbf{w}_{(ij)} \Psi), \quad (\text{no sum over } i). \quad (20)$$

We denote octonions which transform under the same transformation laws as (20) by $\mathbf{O}^{(6)}$. Note that the octonionic transformation factors $\mathbf{w}_{(ij)}$ do not include i_7 . Hereafter, we regard

$$i : = i_7 \quad (21)$$

as the imaginary unit of complex numbers, that is, $A = a_0 + ia_1$ for two real numbers a_0 and a_1 is a complex number.

Proposition 4. $\mathbf{O}^{(6)}$ is a vector space over the complex number field.

This proposition can be proved by confirming that, using the proposition 1, $A(i_{(i,j)} \mathbf{w} \Phi) = i_{(i,j)} (A\Phi)$ holds for an element Φ of $\mathbf{O}^{(6)}$ and a complex number, A , and using the proposition 2, any elements of $\mathbf{O}^{(6)}$ satisfy the axioms of complex vectors, that is, for three elements Φ , Ψ and Ω of $\mathbf{O}^{(6)}$ and two complex numbers A and B ,

$$\begin{aligned}
 (AB) \Phi &= A (B\Phi), \\
 1 \Phi &= \Phi, \\
 0 \Phi &= 0, \\
 (-1) \Phi &= -\Phi, \\
 (A+B) \Phi &= A\Phi + B\Phi, \\
 \Phi + \Psi &= \Psi + \Phi, \\
 \Phi + (\Psi + \Omega) &= (\Phi + \Psi) + \Omega, \\
 A (\Phi + \Psi) &= A\Phi + A\Psi
 \end{aligned}
 \tag{22}$$

We define for two elements Φ and Ψ of $\mathbf{O}^{(6)}$ the following quantity;

$$\{\Phi, \Psi\} := \Phi \cdot \Psi + i (\Phi \cdot (i\Psi))
 \tag{23}$$

Proposition 5. *The quantity $\{\Phi, \Psi\}$ is an invariant complex number under the transformation (20).*

This proposition can be proved by using the formula (10), the proposition 1 and the fact $N(\mathbf{w}_{(i,j)})=1$.

We write an element $\Phi = i_A \phi_A (A=0,1,\dots,7, i_0=1)$ of $\mathbf{O}^{(6)}$

$$\Phi = \Phi_0 + \Phi_2 i_2 + \Phi_3 i_3 + \Phi_6 i_6,
 \tag{24}$$

where

$$\begin{aligned}
 \Phi_0 &= \phi_0 + i \phi_7, & \Phi_2 &= \phi_2 + i \phi_5, \\
 \Phi_3 &= \phi_3 + i \phi_4, & \Phi_6 &= \phi_6 + i \phi_1.
 \end{aligned}
 \tag{25}$$

Then, from the proposition 2, we can show that

$$A\Phi = A\Phi_0 + (A\Phi_2) i_2 + (A\Phi_3) i_3 + (A\Phi_6) i_6
 \tag{26}$$

holds for a complex number A . Moreover, taking another element Ψ of $\mathbf{O}^{(6)}$, and expressing it by the same form as (24), we have

$$\begin{aligned} \{\Phi, \Psi\} &= \Phi_0 \bar{\Psi}_0 + \Phi_2 \bar{\Psi}_2 + \Phi_3 \bar{\Psi}_3 + \Phi_6 \bar{\Psi}_6 \\ &= \overline{\{\Psi, \Phi\}}. \end{aligned} \quad (27)$$

The quantity (27) coincides with the inner product of two vectors in the four dimensional complex unitary space. From (26) and (27), we can prove that

$$\{A\Phi, \Psi\} = \{\Phi, \bar{A}\Psi\} = A \{\Phi, \Psi\} \quad (28)$$

holds for two elements Φ and Ψ of $\mathbf{O}^{(6)}$ and a complex number A . In addition, we can prove also that

$$\{\Phi, (\Psi + \Omega)\} = \{\Phi, \Psi\} + \{\Phi, \Omega\}, \quad (29)$$

hold for three elements Φ , Ψ and Ω of $\mathbf{O}^{(6)}$. Thus, we can identify the invariant complex number (23) of two elements of $\mathbf{O}^{(6)}$ with the inner product of two vectors on which the $SU(4)$ group acts. Note that the isomorphism $SU(4) = Spin(6)$ holds⁹⁾.

Proposition 6. $\mathbf{O}^{(6)}$ is an octonionic representation of the vector space on which the $Spin(6)$ group acts.

We call $\mathbf{O}^{(6)}$ as $Spin(6)$ -octonions. Since the group $Spin(6) = SU(4)$ acts on spinors in \mathbf{R}^6 , we can consider the $Spin(6)$ -octonions as a representation in terms of octonions of the $SO(6)$ spinors.

For two $Spin(6)$ -octonions Φ and Ψ , we consider an octonion product $\Phi\bar{\Psi}$. We can show easily that this product octonion has the same transformation law as \mathbf{V} of (16). However, since $\Phi\bar{\Psi}$ is in general a full octonion, we can decompose it into a sum of a six-octonion with the same form as (15) and the invariant complex number (23). Thus, we can find that for two $Spin(6)$ -octonions Φ and Ψ , the quantity

$$\Phi \cdot (i_i \Psi) = i_i \cdot (\Phi\bar{\Psi} - \{\Phi, \Psi\}) \quad (30)$$

is the component of a vector in \mathbf{R}^6 .

Proposition 7. A component V_i of any vector in \mathbf{R}^6 can be expressed by

$$V_i = \Phi \cdot (i_i \Psi), \quad (31)$$

for two elements Φ and Ψ of $\mathbf{O}^{(6)}$.

The proof of this proposition can be accomplished by putting $\Psi = -\mathbf{V}\Phi/N(\Phi)$ and substituting this into the right hand side of (31), where $\mathbf{V} = i_i V_i$. Note that the

decomposition (31), for a given V_i , is not unique, since for any Φ (or Ψ) there exists an Ψ (or Φ) satisfying(31).

5. Polarization of Spin(6)-octonions

Let us consider a gauge freedom of Spin(6)-octonions. Since, from the proposition 3, Spin(6)-octonions can be regarded as complex vectors, we can consider that these vectors transform under the U(1)-gauge transformation.

Proposition 8-a. *The Spin(6)-invariant complex number $\{ \Phi , \Psi \}$ for two elements Φ and Ψ of $\mathbf{O}^{(6)}$ is invariant under the U(1)-gauge transformation*

$$\Phi \rightarrow \exp(i\Lambda)\Phi, \quad \Psi \rightarrow \exp(i\Lambda)\Psi, \quad (32)$$

where Λ is a real number, and $i=i_i$,

However, we observe that the vector $\Phi \cdot (i_i \Psi)$ changes under the transformation(32). This shows either that a vector in \mathbf{R}^6 cannot be constructed by any two Spin(6)-octonions or that the Spin(6)-octonions cannot have the U(1)-gauge freedoms.

To avoid this difficulty, we have to consider an inequivalent gauge transformation to (32) for elements of $\mathbf{O}^{(6)}$.

Proposition 9. *If two elements Φ and \bar{X} of $\mathbf{O}^{(6)}$ transform under the transformations*

$$\Phi \rightarrow \exp(i\Lambda)\Phi, \quad X \rightarrow X \exp(i\Lambda), \quad (33)$$

respectively, where Λ is a real number, then the vector $\Phi \cdot (i_i X)$ is invariant under these transformations.

We can consider also this transformation for X as a U(1)-gauge transformation for an element of $\bar{\mathbf{O}}^{(6)}$.

Proposition 8-b. *The Spin(6)-invariant complex number $\{ X , \Omega \}$ for two elements X and Ω of $\mathbf{O}^{(6)}$ is invariant under the U(1)-gauge transformation*

$$X \rightarrow X \exp(i\Lambda), \quad \Omega \rightarrow \Omega \exp(i\Lambda), \quad (34)$$

where Λ is a real number.

Therefore, the Spin(6)-octonions are polarized by their U(1)-gauge freedoms, one of the polarizations denoted by $\mathbf{O}^{(6)}$, has the gauge freedoms of the type of (32), and the

other of the polarizations denoted by $\mathbf{O}^{(6)}$, has the gauge freedom of the type of (34), that is,

$$\text{Spin}(6): \Phi \rightarrow \mathbf{i}_i (\mathbf{w}_{(i j)} \Phi), \quad (\text{no sum over } i), \quad (35-a)$$

$$\text{U}(1): \Phi \rightarrow \exp(i\Lambda) \Phi, \quad (35-b)$$

for an element Φ of $\mathbf{O}^{(6)}$, and

$$\text{Spin}(6): X \rightarrow (X \mathbf{w}_{(i j)}) \mathbf{i}_i, \quad (\text{no sum over } i), \quad (36-a)$$

$$\text{U}(1): X \rightarrow X \exp(i\Lambda), \quad (36-b)$$

for an element X of $\mathbf{O}^{(6)}$.

6. Spin(6)-octonions on Six-dimensional Riemannian Manifold

Let us consider the six-dimensional Riemannian manifold K^6 . We denote an orthonormal basis at a point P on K^6 by e_i ;

$$g(e_i, e_j) = \delta_{ij}, \quad (37)$$

where g is a metric tensor at P . Then we can define the Spin(6)-octonions at P , that is, under the following rotation of any (i, j) -plane through an angle θ_P in the tangent space at P ;

$$\begin{aligned} e_i &\rightarrow \cos \theta_P \cdot e_i + \sin \theta_P \cdot e_j, \\ e_j &\rightarrow -\sin \theta_P \cdot e_i + \cos \theta_P \cdot e_j, \\ e_k &\rightarrow e_k, \quad (k \neq i, j) \end{aligned} \quad (38)$$

an element Ψ of $\mathbf{O}^{(6)}$, $(\mathbf{O}^{(6)}$, or $\mathbf{O}^{(6)c}$) at P transforms as

$$\Psi \rightarrow \mathbf{i}_i (\mathbf{w}_{(i j)}(P) \Psi), \quad (\text{no sum over } i), \quad (39)$$

$$\mathbf{w}_{(i j)}(P) = \mathbf{i}_i \cos(\theta_P/2) + \mathbf{i}_j \sin(\theta_P/2) \quad (40)$$

Let us consider the covariant derivative of Spin(6)-octonions. We define the covariant differentiation with respect to the basis e_i at P by ∇_{e_i} . We require that

$$\nabla_{e_i} \Phi = : \nabla_i \Phi, \quad \text{for an element } \Phi \text{ of } \mathbf{O}^{(6)}, \quad (41)$$

are not only elements of $\mathbf{O}^{(6)}$ but also the components of a vector with respect to the

basis e_i at P . Suppose that the proposition 5 and the proposition 6 hold at any point on K^6 . Then we have, for an element Φ of $O^{(6)}$,

$$\nabla_i \Phi = e_i \Phi + \frac{1}{4} \omega_{ijk} (i_j (i_k \Phi)), \tag{42}$$

where ω_{ijk} are the Ricci rotation coefficients¹⁰⁾, which are defined by

$$\nabla_{e_i} e_j = \omega_{ijk} e_k - \omega_{ikj} e_k, \tag{43}$$

and satisfy the torsion-free conditions

$$e_i e_j - e_j e_i = (\omega_{ijk} - \omega_{jik}) e_k. \tag{44}$$

If we consider the polarization of the Spin(6)-octonions, then we have to extend the covariant derivative (42) to be covariant under the local U(1)-transformation. Suppose that the propositions 8-a and 8-b and the proposition 9 hold at any point on K^6 . Then we have, instead of the covariant derivative ∇_i ,

$$D_i \Phi := (\nabla_i + i A_i) \Phi, \text{ for an element } \Phi \text{ of } O^{(6)}_s, \tag{45}$$

$$D_i \bar{X} := (\nabla_i - i A_i) \bar{X}, \text{ for an element } X \text{ of } O^{(6)}_c, \tag{46}$$

where A_i is a gauge field which transforms under the localized transformation of (33) as follows;

$$A_i \rightarrow A_i - e_i \Lambda, \tag{47}$$

and where Λ is a function of the coordinates on K^6 .

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Appendix A : The Octonion Structure Constants

We define the octonion structure constants f_{ABC} ($A, B, C, \dots = 0, 1, \dots, 7$) as follows;

$$i_B i_C := i_A f_{ABC}, \tag{A1-a}$$

where $i_0 = 1$ and i_1, \dots, i_7 are the octonion imaginary units. Then, using (2), the octonion structure constants f_{ABC} can be explicitly written as follows;

$$f_{A0C} = \delta_{AC}, \tag{A2-a}$$

$$f_{ABC} = \delta_{Ab} \delta_{c0} - \delta_{0A} \delta_{bc} + \delta_{Aa} \delta_{Cc} \varepsilon_{abc}$$

$$= -f_{cbA}, (a, b, c, \dots = 1, 2, \dots, 7). \quad (\text{A2-b})$$

The octonion constants f_{ABC} defined by the equations (A1-a) can be also expressed as $f_{ABC} = \mathbf{i}_A \cdot (\mathbf{i}_B \mathbf{i}_C)$. Since, from the identities (10), $\mathbf{i}_A \cdot (\mathbf{i}_B \mathbf{i}_C) = \mathbf{i}_B \cdot (\mathbf{i}_A \bar{\mathbf{i}}_C) = \mathbf{i}_C \cdot (\bar{\mathbf{i}}_B \mathbf{i}_A)$ hold for any octonion units $\mathbf{i}_A, \mathbf{i}_B, \mathbf{i}_C$, then we have

$$\mathbf{i}_A \bar{\mathbf{i}}_C = \mathbf{i}_B f_{ABC}, \quad (\text{A1-b})$$

$$\bar{\mathbf{i}}_B \mathbf{i}_A = \mathbf{i}_C f_{ABC}. \quad (\text{A1-c})$$

Theorem. The octonion structure constants f_{ABC} satisfy the conditions

$$\begin{aligned} f_{(A|B|E|C|D)DE} &= f_{A(B|E|C|D)DE} \\ &= f_{(A|E|B|C|D)ED} = f_{AE(B|C|E|D)} \\ &= f_{E(A|B|C|D)D} = f_{EA(B|C|D)} \\ &= \delta_{AC} \delta_{BD}, \end{aligned} \quad (\text{A3})$$

where the symbol $(A | \dots | B)$ indicates the symmetrization of the indices A and B , i.e. $T_{(A | \dots | B)} = (1/2)(T_{A \dots B} + T_{B \dots A})$.

Proof. Consider $(\mathbf{PQ}) \cdot (\mathbf{PR})$ for three octonions $\mathbf{P}, \mathbf{Q}, \mathbf{R}$. Putting $\mathbf{P} = \mathbf{i}_A p_A$, $\mathbf{Q} = \mathbf{i}_A q_A$ and $\mathbf{R} = \mathbf{i}_A r_A$, and using (A1-a), $(\mathbf{PQ}) \cdot (\mathbf{PR})$ can be written as follows;

$$(\mathbf{PQ}) \cdot (\mathbf{PR}) = f_{EAB} f_{ECD} p_A p_C q_B r_D = f_{E(A|B|E|C|D)} p_A p_C q_B r_D.$$

On the other hand, using (10), we have

$$(\mathbf{PQ}) \cdot (\mathbf{PR}) = \mathbf{R} \cdot (\bar{\mathbf{P}}(\mathbf{PQ})) = N(\mathbf{P})(\mathbf{Q} \cdot \mathbf{R}) = \delta_{AC} \delta_{BD} p_A p_C q_B r_D.$$

Since these equations hold for any p_A, q_A, r_A , thus we have

$$f_{E(A|B|E|C|D)} = \delta_{AC} \delta_{BD}.$$

If we consider $(\mathbf{QP}) \cdot (\mathbf{RP})$, $(\bar{\mathbf{P}}\mathbf{Q}) \cdot (\bar{\mathbf{P}}\mathbf{R})$, $(\mathbf{Q}\bar{\mathbf{P}}) \cdot (\mathbf{R}\bar{\mathbf{P}})$, $(\mathbf{P}\bar{\mathbf{Q}}) \cdot (\mathbf{P}\bar{\mathbf{R}})$ or $(\bar{\mathbf{Q}}\bar{\mathbf{P}}) \cdot (\bar{\mathbf{R}}\bar{\mathbf{P}})$ instead of $(\mathbf{PQ}) \cdot (\mathbf{PR})$, we can prove, using (A1-b,c) instead of (A1-a), the remaining equations. \square

Appendix B: A Correspondence between the Spin(6)-octonions and the Spin(6)-Clifford Algebra

In this appendix, we give a connection between the Spin(6)-octonions and the standard Clifford algebra of the Spin(6) group. For an element Φ of $\mathbf{O}^{(6)}$ and a six-octonion \mathbf{V} constructed by a vector on which the SO(6) group acts, we observe that a product octonion $\mathbf{V}\Phi$ is an element of $\mathbf{O}(6)$. If then we express \mathbf{V} by \mathbf{i}_i

V_i ($i, j, \dots = 1, 2, \dots, 6$) as (15) and Φ by $i_\alpha \Phi_\alpha$ ($\alpha, \beta, \dots = 0, 2, 3, 6$) as (24), the product octonion may be written as follows;

$$\mathbf{V}\Phi = V_i \overline{\Phi}_\alpha (\Gamma_{i\alpha\beta}) i_\beta, \tag{B1}$$

Where

$$\begin{aligned} \Gamma_{i\alpha 0} &:= f_{0i\alpha} + \mathbf{i}f_{7i\alpha}, \\ \Gamma_{i\alpha 2} &:= f_{2i\alpha} + \mathbf{i}f_{5i\alpha}, \\ \Gamma_{i\alpha 3} &:= f_{3i\alpha} + \mathbf{i}f_{4i\alpha}, \\ \Gamma_{i\alpha 6} &:= f_{6i\alpha} + \mathbf{i}f_{1i\alpha}, \end{aligned} \tag{B2}$$

$\mathbf{i} = \mathbf{i}_7$ and f_{ABC} are octonion structure constants.

Proposition 10. We put $\Gamma_{i\alpha\beta}$ defined by (B2) as the (α, β) -component of a 4×4 complex matrix Γ_i , i.e.

$$(\Gamma_i)_{\alpha\beta} := \Gamma_{i\alpha\beta}. \tag{B3}$$

Then the four Γ_i matrices satisfy the Clifford equations

$$\frac{1}{2} (\Gamma_i \Gamma_j^+ + \Gamma_j \Gamma_i^+) = \delta_{ij} I, \tag{B4}$$

where the symbol $+$ indicates the hermite conjugation of a matrix and I is the 4×4 unit matrix.

Proof. Consider an invariant quantity $\{\mathbf{V}\Phi, \mathbf{V}\Psi\}$ for two spin(6)-octonions Φ, Ψ and a six-octonion \mathbf{V} . From (24), (B1) and (B3), we have

$$\begin{aligned} \{\mathbf{V}\Phi, \mathbf{V}\Psi\} &= (\mathbf{V}\Phi)_\alpha \overline{(\mathbf{V}\Psi)_\alpha} \\ &= V_i V_j \Phi_\alpha (\Gamma_{(i} \Gamma_{j)}^+)_{\alpha\beta} \Psi_\beta, \end{aligned}$$

where the symbol (...) in the indices indicates the symmetrization. On the other hand, from (23), (B1) and (B2), we have

$$\begin{aligned} \{\mathbf{V}\Phi, \mathbf{V}\Psi\} &= (\mathbf{V}\Phi) \cdot (\mathbf{V}\Psi) + \mathbf{i}((\mathbf{V}\Phi) \cdot (\mathbf{i}((\mathbf{V}\Psi)))) \\ &= N(\mathbf{V}) (\Psi \cdot \Phi + \mathbf{i}(\Psi \cdot (\mathbf{i}\Phi))) \\ &= N(\mathbf{V}) \{\Psi, \Phi\} \\ &= V_i V_j \delta_{ij} \Phi_\alpha \Psi_\alpha. \end{aligned}$$

Since these two equations hold for any Φ_α, Ψ_α and V_i , we obtain the equation (B4). \square

The Clifford equations (B4) are obviously of the Spin(6) group, namely, Γ_i are the gamma matrices in the six-dimensional Euclidean space. Thus, we can show the corrective correspondence between the Spin(6)-octonions and the Spin(6)-Clifford algebra.

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