On the Quantum Theory of Gravity by the Formulation of the Background Field Theory.

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Summary : The quantum theory of gravitation was developed on the basis of the background field formulation originally developed by Feynman to quantize a perturbation field of the classical gravitational field. The theory is, therefore, intrinsically semiclassical picture of the graviation.

In this paper, we present a connection between the two important formulations of the quantum theory of gravity : the original formulation of DeWitt's background field method and the path integral method developed by Hawking.

I. Introduction.

Recently, the background field theory has called attention of physicists as a new method of quantization of not only Yang-Mills fields but also a gravitational field.

The fundamental idea of this background field theory was first put forward by Feynman(1) and was formulated by DeWitt(2). Since then, many authors discussed from various view points(3). In the case of Yang-Mills fields, the gauge invariance has to be implemented into the formulation as a fundamental requisit. In the case of gravitation, as for the background field, we know that the correct classical gravitation is that of Einstein. The general theory of relativity is constructed on the basic principle that the theory should be invariant under general coordinate transformations. It is important, therefore, that in the case of quantization of the gravitational field, the whole process of the quantization of the gravitational field should be performed in a manifestly invariant way under general coordinate transfrmations as in the case the quantization of Yang-Mills field is carried out by the gauge invariant way. The background field theory utilized this idea in a maximum extent.

Recently, an attempt is made that the theory is to satisfy the BRS symmetry(4). The formulation of the background field theory differs from that of the usual quantum field theory.

The theory is constructed using the S matrix. Consider a classical field ϕ is perturbed infinitesimally $\phi = \phi_0 + \phi'$. The perturbed field ϕ' is considered as a quantum fluctuation and the field ϕ' is quantized while the unperturbed field ϕ_0 :

the "background field is considered as a classical field. The perturbed field is expanded around the background field ϕ_0 and obtain the S matrix. Thus, this quantum mechanical formulation of the gravitational field is, inevitably a semiclassical picture. The reason for this is that a complete quantization of a classical gravitational field is, at present, extremely difficult, as many authors pointed out. It seems that the path integral method is the best for the quantization of a gravitational field. The path integral method of the quantization of the gravitational field were introduced by S. W. Hawking and G. W. Gibbons[5], but a rigorous proof of the validity of the path intergral method has not been obtained.

In this paper, an attempt is made to transfer from the original DeWitt formulation of quantization of a gravitational field by the background field method to the path integral method of Hawking.

In Section 2, the original DeWitt formulation is introduced and in Section 3, a method to transfer to the path integral method of Hawking is described.

2. Original Formulation of DeWitt.

In this Section we introduce the original formulation of the S matrix theory. The vacuum-to-vacuum transition matrix elements have the following form:

$$<0,\infty \mid 0,-\infty> = e^{i\Gamma} \tag{1}$$

In the above eq. (1) , the righr-hand side is the functional integral defined by

$$e^{i\Gamma} = N\exp[\, i(\, S + \frac{1}{2\,!} S_{,ij} \tilde{\varphi}^i \tilde{\varphi}^j + \frac{1}{3\,!} S_{,ijk} \tilde{\varphi}^i \tilde{\varphi}^j \tilde{\varphi}^k + \cdots)\,] \tag{2}$$

where S is the classical action. The notation of DeWitt is used hereafter :

$$\varphi^{i'} \equiv \varphi^i(x')$$
$$F_{,'\mu\nu\cdots} = \cdots\cdots \frac{\partial}{\partial x'^{\mu}} \frac{\partial}{\partial x'^{\nu}} F$$

For δ -function :

$$\delta^{i'}_j \equiv \delta^i_j \delta(x,x'),$$
$$\delta^{(\mu\nu)}_{(\sigma'\tau')} \equiv \frac{1}{2}(\delta^{\mu}_{\sigma}\delta^{\nu}_{\tau} + \delta^{\mu}_{\tau}\delta^{\nu}_{\sigma})\delta(x,x'),$$
$$\delta^{[\mu\nu]}_{[\sigma'\tau']} \equiv \frac{1}{2}(\delta^{\mu}_{\sigma}\delta^{\nu}_{\tau} - \delta^{\mu}_{\tau}\delta^{\nu}_{\sigma})\delta(x,x')$$

The functional derivatives $S_{,ij}$, $S_{,ijk}$ are those of differentiation by the background field operators δ^i. Eq. (2) is equivalent to the action in which the field operators are replaced by

$$\varphi^{i} \to \varphi^{i} + \tilde{\varphi}^{i} \tag{3}$$

where the δ^{i} are the classical background field, and the δ^{i} are the perturbation field.

Since the equation (2) does not take into account the ultraviolet divergence, the expression differs slightly in form from that given by G. t'Hooft[6].

In eq. (2), the first functional derivatives do not appear because the background field satisfies the classical field equations : $S_{,i} = 0$.

The diagrams constructed using (2), are the loop expansion in case where there does not exist an infinite dimensional invariance group to the background field.

The states of the system are represented on the mass shell states. The expansion in (2) is equivalent to that by the Planck's constant h [7].

In case where there is an infinite dimensional invariance group, the diagrams are replaced by the two loops, the one expresses the propagator of the physical quantity on mass shell and the other does that of fictitious quantum. As for example, in the case of a grtavitational field, the quantum is spin one and the physical quantity describing the propagation is expressed by the following form :

$$\mathrm{In}\left[\frac{\det(G^{ij})\det(G_0^{\alpha\beta})^2\det(G_0^{-ij})\det(G^{-\alpha\beta})^2}{\det(G_0^{ij})\det(G^{\alpha\beta})^2\det(G^{-ij})\det(G_0^{-\alpha\beta})^2}\right] \tag{4}$$

where $G^{\alpha\beta}$ are the propagator for the operator F_{ij} of the virtual particle and $G_0{}^{\alpha\beta}$ are the propagator obtained by the calculation when there exists no perturbation field and G^{-ij} and $G^{-\alpha\beta}$ are the retarded Green functions of F_{ij} and $F_{\alpha\beta}$, respectively.

The $G^{\alpha\beta}$ are, in general, the Green function of the following operators :

$$F_{\alpha\beta} = R^{i}_{\alpha}\gamma_{ij}R^{i}_{\beta} \tag{5}$$

where $R^{i}{}_{\alpha}$ are the functions which appear in the infinitesimal group transformations of the field quantities φ^{i} :

$$\delta\varphi^{i} = R^{i}_{\alpha}\delta\xi^{\alpha} \tag{6}$$

where ξ^{α} are the group parameters. γ_{ij} in eq. (5) are the elements of the symmetric continuous matrices. A concrete form of the infinitesimal disturbance can be determined by an auxiliary condition. Furthermore, the R^{i} satisfy the following identities :

$$R^{i}_{\alpha,j}R^{j}_{\beta} - R^{i}_{\beta,j}R^{j}_{\alpha} = R^{i}_{\gamma}C^{\gamma}_{\alpha\beta} \tag{7}$$

where $C^{\gamma}{}_{\alpha\beta}$ are the structure constants.

In the case of a gravitational field and a Yang-Mills field, $S_{,ij}$ in eq. (2) are singular. In the field theory, however, those quantities are requested to be non-singular. To meet the requirement, $S_{,ij}$ are replaced by F_{ij} defined by

$$F_{ij}=S_{,ij}+\gamma_{ik}R^{k}_{\alpha}\gamma^{\alpha\beta}R^{l}_{\beta}\gamma_{lj} \tag{8}$$

where $\gamma^{\alpha\beta}$ are nonsingular symmetric matrices and obey the following group transformation law :

$$\delta\gamma^{\alpha\beta}\equiv\gamma^{\alpha\beta}{}_{,i}R^{i}_{\gamma}\delta\xi^{\gamma}=(C^{\alpha}_{\gamma\delta}\gamma^{\delta\beta}+C^{\beta}_{\gamma\delta}\gamma^{\alpha\delta})\delta\xi^{\gamma} \tag{9}$$

Let the action for Einstein theory be taken as

$$S_{g}=-\int g^{\frac{1}{2}}R\,dx-\lambda\int g^{\frac{1}{2}}dx \tag{10}$$

where R and λ are Riemann scalar and cosmological constant, respectively, and we take the units : $16\pi G=c=1$.

Using eq. (10), the $S_{,ij}$ in eq. (2) are calculated to be as follows[8] :

$$\delta^{2}S_{g}/\delta g_{\mu\nu}\delta g_{\sigma'\tau'}=$$

$$=\left\{\begin{array}{l}\frac{1}{4}g^{\frac{1}{2}}(g^{\mu\rho}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\rho\lambda})\times[g^{lk}(\delta^{(\sigma'\tau')}_{(\rho\lambda)\cdot lk}\\ +\delta^{(\sigma'\tau')}_{(lk)\cdot\rho\lambda}-\delta^{(\tau'\tau')}_{(l\rho)\cdot\lambda k}-\delta^{(\sigma'\tau')}_{(\lambda k)\cdot\rho l})+\lambda\,\delta^{(\sigma'\tau')}_{(\rho\lambda)}]\end{array}\right. \tag{11}$$

where the subscript dotts in the δ function denote the covariant derivatives.

Further, choosing the following supplementary conditions :

$$\delta g_{\mu\nu}=\int R_{\mu\nu\sigma'}\delta\xi^{\sigma'}dx' \tag{12}$$

$$R_{\mu\nu\sigma'}\equiv-\delta_{\mu\sigma'\cdot\nu}-\delta_{\nu\sigma'\cdot\mu} \tag{13}$$

$$\delta_{\mu\nu'}\equiv g_{\mu\nu}\delta(x,x')$$

and let the matrices be defined as

$$\gamma^{\mu\nu\sigma'\tau'}\equiv-\frac{1}{2}g^{\frac{1}{2}}(\delta^{(\mu\nu)(\sigma'\tau')}-\frac{1}{2}g^{\mu\nu}\delta^{\rho)\sigma'\tau'}_{(\rho}) \tag{14}$$

$$\gamma^{\mu\nu'}\equiv\delta^{\mu\nu}g^{-\frac{1}{2}}$$

the $F^{\mu\nu\sigma'\tau'}$ corresponding to the F_{ij} are expressed as[8]

$$\begin{aligned}F^{\mu\nu\sigma'\tau'}&=\frac{1}{2}g^{\frac{1}{2}}(g^{\mu\rho}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\nu\lambda})\\&\times(\delta^{(\sigma'\tau')k}_{(\rho\lambda)\cdot k}-2R_{\rho l\lambda k}\delta^{(lk)(\sigma'\tau')})\end{aligned} \tag{15}$$

$F^{\mu\upsilon\sigma'\tau'}$ given by (15) are non-singular self-adjoint operators and the Green functions can be defined.

3. The relation of the path integral formulation by Hawking to the original DeWitt formulation.

To consider a quantization of gravitation, Hawking proposed the following path integral form for the amplitude :

$$Z=\int D[g]D[\phi]\exp\{iI[g,\phi]\} \tag{16}$$

where D[g] is a measure on the space of all metrices, $D[\phi]$ is a measure on the space of all matter fields, I is the action and the integral is taken over all field configurations with a given initial and final values of ϕ and g.

The integral Z is the amplitude $<g_2,\phi_2,S_2 \mid g_1,\phi_1,S_1>$ to go from a state $\mid lg_1,\phi_1, S_1>$ with metric g_1, matter field ϕ_1 on the S_1 surface at t_1 to a state $\mid lg_2, \phi_2, S_2>$ with metric g_2, matter field ϕ_2 on the S_2 surface at time t_2 :

$$<g_Z,\phi_2,S_2 \mid g_1,\phi_1,S_1>=Z \tag{17}$$

If the amplitude (17) can be determined by specifying the background fields g_1, ϕ_1, S_1 and g_2, ϕ_2, S_2, then $\tilde{\varphi}^i$ in eq. (3) are regarded as field quanta.

In the case of a gravitational field, eq. (3) has to be replaced by

$$g_{\mu\nu}=\overset{c}{g}_{\mu\nu}+\tilde{g}_{\mu\nu} \tag{18}$$

where $\tilde{g}_{\mu\nu}$ are the field arising from the quanum fluctuation of the classical background field $\phi^c{}_{\mu\nu}$.

The classical background field bounded by the two space-like surfaces is essentially "compact". The background field plays a part of the source of the perturbed field. In case where there is an infinite dimensional invariant group, the order of the operators in a product of operators cannot, in general, be determined uniquely so that it is difficult to introduce a source term in the theory. Thus, to obtain an interaction into the theory, we have to use the perturbation expansion based on the relation given by eq. (18).

Let $\lambda=0$ in eq. (10), the action integral is given as

$$S_g=-\int g^{\frac{1}{2}}R\,dx \tag{19}$$

The Ricci scalar R contains terms of the second derivatives of the metrices.

In order to obtain an action which depends only on first derivatives, as is required by the path integral method, one has to remove the second derivatives by integration by parts. We obtain for the acation I, instead of eq. (19), as follows :

$$I=-\int_Y Rg^{\frac{1}{2}}d^4x+\int_{\partial Y}B(-h)^{\frac{1}{2}}d^3x \tag{20}$$

where Y is a region bounded by the space-like surfaces S_1 and S_2 and δY is the boundary of the region Y as in the case of S_1 and S_2 described in eq. (17) .

The region Y is a compact region and in this region, the background field is asymptotically flat.

The surface term B is so chosen as to that the metric g satisfies Einstein field equation and h is the induced metric on the boundary.

The action I is, for the variation of the metric g (such as due to a gauge transformation), zero on the boundary surface δ Y.

The surface term B, in general, ignoring the coefficient term, has to be of the following form :

$$B = K + C \tag{21}$$

where K is trace of the second fundamental form of the boundary of the metric g depends on the induced metric h and C is a term which depends only on the boundary and not on a particular metric g.

Using S_g given by eq. (19), $S_{,ij}$ of (11) is given as

$$\delta^2 S_g \diagup \delta g_{\mu\nu}\, \delta g_{\sigma',\tau'}$$

$$= \left\{ \begin{array}{l} \frac{1}{4} g^{\frac{1}{2}} (g^{\mu\rho} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\rho\lambda}) \times \\ [g^{lk} (\delta^{(\sigma'\tau')}_{(\rho\lambda)\cdot lk} + \delta^{(\sigma'\tau')}_{(\rho l)\cdot \lambda k} - \delta^{(\sigma'\tau')}_{(\lambda k)\cdot \rho l})] \end{array} \right. \tag{22}$$

The second fundamental derivatives derived from the action given by eq.(20) do not in general, agree with those given by eq. (22).

It is, however, a difficult problem, even when the absence of the matter field ϕ^i, whether non-singular self-adjoint operators $F^{\mu\nu\sigma'\tau'}$ which correspond to those of eq. (15) can be derived from eq. (20) under certain supplementary conditions as those of eq.(12) and eq. (13). The second term on the right hand side of eq. (20) is of the same form as that given by eq. (21) in a finite region of the background field. In this case, F_{ij} given by eq. (8) have Green functions.

The K given in eq. (21) is, in general, the second fundamental form so that Gauss' theorem holds on any chosen closed surface. In general, an arbitrary differential operator F_{ij} should have Cauchy data, and the Gauss' theorem should hold on such a closed surface. The second term on the right hand side of eq. (20), is invariant under a general coordinate transformation of the boundary δ Y of a finite region of the background field.

In order that the theory constructed on the basis of the path integral defined by eq. (16) should be "manifestly covariant", the asymptotic conditions[2] for the background field should not change their forms under general coordinate transformations because the background fields introduced in Sections (2) and (3) should essentially be the same field.

4. Conlusion.

In the above derivations of the path integral methed and the original DeWirtt formulation, the procedures are not rigorous from the mathematical view point.

There arises an intrinsic difficulty to define a Feynman path integral in the definition of the path integral in the functional space of the metrices in a flat Minkowski space. The difficulty is more profound in the case of a gravitational field. [The region Y on the background field introduced in Section 3 is asymptotically flat. But if it were not asymptotically flat, the time slice of the space-time of the path of the background field cannot be made uniform. The infinitesimal perturbation can be given only in such a spacetime in which a uniform slice is to be made possible.]

If the argument developed in Section 3 should hold rigorously, then apart from the "manifestly covariance" of the theory, the g field defined in eq. (18) can be considered to be a "quantized field".

The difficulty to introduce in Lagrangian formalism the source terms of the gravitational field, the expression to describe the interactions of the gravitational fields corresponding to those of eq. (2) can be obtained by the substitution of the $g_{\mu\nu}$ given by eq. (18) into those of eq. (16).

A further study is going on at the moment and the full result will be given in future.

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