On the Relevant Transforms and Prime Divisors of the Powers of an Ideal

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0. Throughout this paper the following notation is fixed. Let R be a Noetherian integral domain. Let t be an indeterminate, and let I be a non-zero ideal of R. Let denote $A_I = R[t^{-1}, tI]$ the *Rees ring of R with respect to* I. Further, after Mirbagheri and Ratliff (4), we call the ideal $I^* = \cup \{I^{i+1} :_R I^i ; i \geqq 1\}$ the *relevant component of* I, and the ring $B^I = R[t^{-1}, \{t^i(I^i)^*\}_{i \leqq 1}]$ the *relevant thansform of* I. Thus both rings A_I and B_I are subrings of the *torus extension* $T_R = R[t^{-1}, t]$ *of* R. Let $A^*(I)$ denote the set $\{p \in \mathrm{Spec}(R) ; p \in \mathrm{Ass}_R(R/I^n)$ for all large $n\}$. Our general reference for undefined terminology is (2).

1. We first state some known results about $A^*(I)$ and I^*.

(1.1) It was shown by Brodmann that $A^*(I)$ is a well-defined finite set (cf.(3)Corollary 1.5).

(1.2) In our case when R is an integral domain, $A^*(I) = \{P \cap R ; P$ is a prime divisor of $t^{-1} A_I$ not containing $tI\}$ (3) Propositions 1.15 and 2.2.

(1.3) It is known (5) that (a) $I^* = I^{n+1} :_R I^n$ for all large n, (b) $(I^h)^* = \cup \{I^{h+i} :_R I^i ; i \geqq 1\}$, and (c) $(I^h)^* = I^{h+n} :_R I^n$ for all large n.

(1.4) Also, (5)shows that $(I^n)^* = I^n$ for all large n.

2. In this section we shall prove several properties of the relevant transform of an ideal.

Proposition 2.1. The relevant transform B_I *of* I *is the greatest subring* C *between* A_I *and* T_R *satisfying the condition that* $(tI)^N C$ *is contained in* A_I *for some positive integer* N. *In particular,* B_I *is a finite* A_I*-module* (cf.(4)(2.4)).

Proof. It follows from (1.3) and (1.4) that for some positive integer H, $(I^i)^* = I^i$ for all $i \geqq H$, and for some positive integer N, $(I^i)^* = I^{i+N}$ for all $i < H$.

Then $(tI)^N (I^i)^* t^i = I^N (I^{i+N} :_R I^N) t^{i+N} \subseteq {}^{i+N} \subseteq I^{i+N} t^{i+N}$ for all $i < H$, and obviouly $(tI)^N (I^i)^* t^i = I^{i+N} t^{i+N}$ for all $i \geqq H$. Thus $(tI)^N B_I \subseteq A_I$. Now, let C be a subring between A_I and

T_R that satisfies $(tI)^N C \subseteq A_I$ for some positive integer N. Let $c \in C$. In order to see $c \in B_I$ we may take $c = at^i$ with $a \subseteq R$ and a positive integer i. Then $I^N a \subseteq I^{i+N}$, and so $a \in I^{i+N}:_R I^N$. hence $a :\in (I^i)^*$ by (1.3). Thus $C \subseteq B_I$.

Another characterization of B_I can be given in terms of the *idealizers* $Id((tI)^n)$ *of the ideal* $(tI)^n A_I$ *in* T_R, defined by $Id((tI)^n) = (tI)^n A_I :_{T_R} (tI)^n A_I$.

Theorem 2.2. There exists a finite sequence of idealizers $Id((tI)^n)$ *from* A_I to B_I *such that* $A_I \subseteq Id(tI) \subseteq ... \subseteq Id((tI)^N) = B_I$.

Proof. It is enough to see that $Id((tI)^n) = A_I :_{T_R} (tI)^n A_I$ because then the theorem will follow from (2.1). Now one inclusion is obvious, so we have to verify that $Id((tI)^n) \supseteq A_I : T_R((tI)^n)$. To do so, we may take $b \in T_R$ with $b = at^i$, where $a \in R$ and i is a positive integer, such that $b(tI)^n A_I \subseteq A_I$. Then one can see easily that $b(tI)^n \subseteq I^{i+n} t^{i+n} \subseteq (tI)^n A_I$. Hence $b \in Id((tI)^n)$. This establishes $Id((tI)^n) = A_I :_{T_R} (tI)^n A_I$, and the proof is complete.

Corollary 2.3. $B_I = A_I$ *if and only if* $Id(tI) = A_I$.

Proof. One direction is immediate from (2.2). So we assume that $Id(tI) = A_I$. Then the proof of (2.2) shows that $Id(tI) = A_I :_{T_R} tIA_I$, and so $A_I = A_I :_{T_R} tIA_I$. Now, $b \in B_I$ satisfies $b(tI)^n A_I \subseteq A_I$ for some positive integer n. Then $b(tI)^{n-1} A_I \subseteq A_I$. Therefore by induction on the powers n, we obtain $b(tI)A_I \subseteq A_I$. Hence $b \in A_I$. Thus $B_I = A_I$.

3. This section contains a characterization of the set $Ass_{A_I}(B_I/A_I)$ and a theorem for prime divisors of some I^n which are not contained in $A^*(I)$. To prove these, we need a notion: for a non-positive integer n, the power I^n always means R.

Theorem 3.1. For a prime ideal P *of* A_I,*the following statements are equivalent.*

(3.1.1) P *is a prime divisor of* $t^{-1} A_I$ *containing* tI.

(3.1.2) $P \in Ass_{A_I}(B_I/A_I)$.

Proof. Assme that (3.1.1) holds. So $P = t^{-1}A_I : A_I b$ for some $b \in A_I$. Now write $b = \sum_{i=e}^{n} a_i t^i$ *,where* $a_i \in I_i$ $e \leqq i \leqq n$. *Since* $tI \subseteq P$, *it follows that* $a_i I \subseteq I^{i+2}$, *therefore* $a_i \subseteq I^{i+2} :_{RI} I \overset{i+1}{\subseteq} (I^{i+1})^*$ *by (1.3). Thus* $bt \in B_I$, *and so (3.1.2) holds. Conversely, if (3.1.2) holds, then* $P = A_I :_{AI} at^n$ *for some* $a \in R[t^{-1}]$ *and some positive integer n. This clearly shows* $depth(A_I)_P = 1$. *Now if* $t^{-1} \notin P$, *then* $P = A_I :_{AI} at^n = A_I$. *So we see that P contains* t^{-1} *which in turn, shows that P is a pwime divisor of* $t^{-1} A_I$. *It remains to see that tI is contained in P. But this is a direct consequence of (2.1). Hence (3.1.2) implies (3.1.1). The proof is complete.*

Theorem 3.2. Let $p \in Ass_R(R/I^n)$ for some n. If $p \notin A^*(I)$, Then $p = P \cap R$ for $P \in Ass_{A_I}(B_I/A_I)$.

Proof. Let $p \in Ass_R(R/I^n)$. Since $t^{-n} A_I \cap R = I^n$, we can find a prime divisor P of $t^{-n} A_I$ such that $P \cap R = p$. Now $p \notin A^*(I)$, therefore $P \supseteq tI$ by (1.2). Thus P satisfes (3.1.1), and so $P \in Ass_{AI}(B_I/A_I)$, completing the proof.

4 . We shall investigate the graded R/I-algebra $G_R(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ to extend some results of Robbiano and Valla [6] concerning the primary of a prime ideal to the prime divisors of I^n.

Proposition 4.1. Let N *be a positive integer. The following statement holds.* $\mathrm{Ass}(R/I^n) \subseteq \mathrm{Ass}_R(R/I)$ *for all* $n \leqq N$ *if and only if* I^{n-1}/I^n *is a torsion free* R/I-*module for all* $n \leqq N$.

Proof. Assume that $\mathrm{Ass}_R(R/I^n) \subseteq \mathrm{Ass}_R(R/I)$. Let x be an element of R not contained in any member of $\mathrm{Ass}_R(R/I)$. Therefore the image of x in R/I is a non-zero-divisor. Let $y \in I^{n-1}$ satisfy $xy \in I^n$. Then $y \in I^n$. Thus I^{n-1}/I^n is a torsion free R/I-module. To see the converse, we assume that I^{n-1}/I^n is a torsion free R/I-module for all $n \leqq N$. Let $p \in \mathrm{Ass}_R(R/I^n)$, and then localize R at p. Hence R is a local ring with maximal ideal p. Now suppose that $p \notin \mathrm{Ass}_R(R/I)$. Then we can find an element x of p not contained in any member of $\mathrm{Ass}_R(R/I)$. Since p is a prime divisor of I^n, there exists an element $y \in R$ such that $y \notin I^n$ and $yp \subseteq I^n$. Therefore $xy \in I^n$. Let r be a non-negative integer less than n such that $y \in I^r$ but $y \notin I^{r+1}$. Since I^r/I^{r+1} is a torsion free R/I-module, $xy \notin I^{r+1}$.This contradicts $xy \in I^n \subseteq I^{r+1}$. We are done. *Theoren 4.2. The following statements hold.*

(4.2.1) $G_R(I)$ *is a torsion free* R/I-module if and only if $\cup_{n\geq 1} \mathrm{Ass}_R(R/I^n) = \mathrm{Ass}_R(R/I)$. *In particular, in this case* $A^*(I) \subseteq \mathrm{Ass}_R(R/I)$.

(4.2.2) (cf.(6) Corollary 1.2) *Let* p *be a prime ideal and* I *be a* p-*primary ideal of* R. *Then* $G_R(I)$ *is a torsion free* R/I-module if and only if I^n *is a* p-*primary ideal for every positive integer* n.

If $I^* = I$, then $I = I^{n+1} :_R I^n$ for all positive integers n, or equivalently, $\mathrm{Ann}_R(I^n/I_{n+1}) = I$ for all positive integers n. Thus we see that $\mathrm{Ass}_R(R/I) = \mathrm{Ass}_R(I^n/I^{n+1})$ for all non-negative integers n. We combine this with (4.2.1) to obtain the following.

Corollary 4.3. If $I^* = I$ *and* $G_R(I)$ *is a torsion free* R/I-*module, then* $\mathrm{Ass}_R(R/I) = \mathrm{Ass}_R(R/I^n)$ *for all non-negative integers* n.

5 . To close the paper, we give remarks related to (2.3). A sufficient condition for $A_I = B_I$ is that A_I is seminormal (see, for seminormality [1]). To see this, we suppose that $A_I \neq B_I$. Then we have $I^n \neq (I^n)^*$ for some positive integen n. Let a be an element of $(I^n)^*$ not contained in I^n. By (1.3), $a^h \in I^{nh}$ for all large h, but $at^n \notin A_I$. Thus A_I is not seminormal. Next we show by way of a specific example that the converse of the above staement is false: Let k be a field and x be an indeterminate. Let $R=k[x^2,x^3]$, and $I = x^2R$. Evidently R is not seminormal, hence the Rees ring $A_I = k[t^{-1},x^2t, x^3]$ is not seminormal. It is easy to see that $(I^n)^* = I^n$ for positive integer n, and so $A_I = B_I$.

The argument made above also shows that B_I is contained in the seminormalization of A_I. We think that it might be intereating to determine or to characterize the seminormalization of A_I. But we have not any information about this.

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