Octonions and a Null Cone in Ten-dimensional Minkowski Space-time

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Abstract

A geometrical association between octonions and a null cone in the ten-dimensional Minkowski space-time is discussed. A point on the intersection of the null cone with a spatial hyperplane can be mapped to a point on the octonionic Gauss plane by a stereographic projection. Then, we may see that a null direction in the ten-dimensional Minkowski space-time corresponds to an octonion. Identifying such an octonion as an ($SO(8)-$) vectorial octonion, left- and right-acted spinorial octonions are defined. A connection between these three octonions is aregued.

1. Introduction

One of the recent developments of mathematical physics revealed that the space-time is ten-dimension. Particularly, by great success of the superstring theory[1], one believes that the space-time dimension of the era at which all the forces were unified is certainly "ten". Namely, in only the ten-dimension, we have a consistent unified theory.

Kugo and Townsend[2] studied an association of the assosiative division algebras with space-time dimensions by a systematic method. They obtained the association of isomorphisms $SL(2, \boldsymbol{R}) \cong SO(2, 1)$, $SL(2, \boldsymbol{C}) \cong SO(3, 1)$ and $SL(2, \boldsymbol{H}) \cong SO(5, 1)$, where $\boldsymbol{R}$ is real numbers, $\boldsymbol{C}$ complex numbers and $\boldsymbol{H}$ quaternions, and $SO(2, 1)$ is the symmetry to three-, $SO(3, 1)$ four- and $SO(5, 1)$ six-dimensional Minkowski space-time. Although they had guessed that octonions which is one of division algebras may be associated with the ten-dimensional Minkowski space-time, this could not be proved because of the nonassociativity of octonions.

This problem was solved by Davies and Joshi[3]. They were proved, by a bimodular representaion[4], that octonions associate with the ten-dimensional Minkowski space-time. Further, more recently, Evans established an explicit correspondence between simple super-Yang-Mills and classical superstrings in the Minkowski space-time of dimensions 3, 4, 6, and 10 and real numbers, complex numbers, quaternions and octonions, respectively[5]. Thus, we believe that octonions play a fundamental role in the ten-dimensional Minkowski space-time.

In this paper, we describe only the association between the ten-dimensional Minkowski space-time and the octonions, since we are interested in the ten-dimension, only. Moreover, we are not only interested in the algebraic association but also a geometric one. An octonion naturally corresponds to a null direction in the ten-dimensional Minkowski space-time. Namely, the geometry with which we deal is a null cone geometry.

2. Algebra of Octonions

In this section, we briefly describe an algebra of octonions denoted by $\boldsymbol{O}$[6]. We define $\boldsymbol{a} \in \boldsymbol{O}$ as

$$\boldsymbol{a} = a_0 + \boldsymbol{i}_1 a_1 + \ldots\ldots + \boldsymbol{i}_7 a_7, \tag{2-1}$$

where $\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots\ldots, \boldsymbol{i}_7$ are octonion imaginary units having the relations

$$\begin{aligned} \boldsymbol{i}_A \boldsymbol{i}_B &= -\delta_{AB} + \boldsymbol{i}_C \varepsilon_{ABC},\ (A,\ B,\ C = 1,\ 2,\ \ldots\ldots,\ 7), \\ \varepsilon_{ABC} &= \varepsilon_{[ABC]} = 1, \text{ for } ABC = 123,\ 145,\ 176,\ 246,\ 257,\ 347,\ 365. \end{aligned} \tag{2-2}$$

And we define the octonion conjugate of $\boldsymbol{a}$ as

$$\bar{\boldsymbol{a}} = a_0 - \boldsymbol{i}_1 a_1 - \ldots\ldots - \boldsymbol{i}_7 a_7. \tag{2-3}$$

Then, for $\boldsymbol{a},\ \boldsymbol{b} \in \boldsymbol{O}$, we have an equation

$$\overline{\boldsymbol{ab}} = \bar{\boldsymbol{b}}\bar{\boldsymbol{a}}. \tag{2-4}$$

This algebra is not only noncommutative but also nonassosiative, that is, for $\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in \boldsymbol{O}$,

$$[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}] = (\boldsymbol{ab})\boldsymbol{c} - \boldsymbol{a}(\boldsymbol{bc}) \neq 0. \tag{2-5}$$

We define the inner product of $\boldsymbol{a},\ \boldsymbol{b} \in \boldsymbol{O}$ as

$$\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = \frac{1}{2}(\boldsymbol{a}\bar{\boldsymbol{b}} + \boldsymbol{b}\bar{\boldsymbol{a}}) = \frac{1}{2}(\bar{\boldsymbol{a}}\boldsymbol{b} + \bar{\boldsymbol{b}}\boldsymbol{a}). \tag{2-6}$$

Then, with another $\boldsymbol{c} \in \boldsymbol{O}$, we may prove

$$\langle \bar{\boldsymbol{a}},\ \boldsymbol{bc} \rangle = \langle \bar{\boldsymbol{b}},\ \boldsymbol{ca} \rangle = \langle \bar{\boldsymbol{c}},\ \boldsymbol{ab} \rangle. \tag{2-7}$$

The norm of $\boldsymbol{a}$ is denoted by its self inner product as follows :

$$N(\boldsymbol{a}) = \langle \boldsymbol{a},\ \boldsymbol{a} \rangle. \tag{2-8}$$

Note that the following equation colds for $\boldsymbol{a},\ \boldsymbol{b} \in \boldsymbol{O}$

$$N(\boldsymbol{ab}) = N(\boldsymbol{a})N(\boldsymbol{b}). \tag{2-9}$$

Thus, octonions are the division algebra.

3. Future Null Cone in Ten-dimensional Minkowski Space-time and Octonionic Gauss Plane

Let us consider a future null cone N^+ with a light source at the origin in the ten-dimensional Minkowski space-time M^{10} with the coodinate $(t,\ x,\ y_1,\ \ldots\ldots,\ y_7,\ z)$, which the metric is defined by

$$\eta^{ab} = \mathrm{diag}(1,\ -1,\ \ldots\ldots,\ -1),\ (a,\ b = 0,\ 1,\ \ldots\ldots,\ 9). \qquad (3\text{-}1)$$

We denote the intersection of N^+ with a spatial hyperplane $t = 1$ by S^+ which is equivarent to an 8-sphere with unit radius. Namely, a point $(1,\ \overset{0}{x},\ \overset{0}{y}_1,\ \ldots\ldots,\ \overset{0}{y}_7,\ \overset{0}{z})$ on S^+ satisfies

$$(\overset{0}{x})^2+(\overset{0}{y}_1)^2+\ \ldots\ldots\ +(\overset{0}{y}_7)^2+(\overset{0}{z})^2 = 1. \qquad (3\text{-}2)$$

Now, we consider a vector V^a passing through the origin O. If V^a is a future time-like vector, then the intersection of the straight line along V^a with the hyperplane $t = 1$ is inside S^+, and if V^a is a space-like vector, then te intersection is outside S^+. Futhermore, if V^a is a future null vector, then the intersection is on S^+. Thus, the direction of (a future caunal and space-like vector) V^a may be determined by a point on the hyperplane $t = 1$; $(V^1/V^0,\ V^2/V^0,\ \ldots\ldots,\ V^9/V^0)$.

Introduce a null vector L^a passing through the origin O, and take its coodinate components as follows ;

$$L^a = (T,\ X,\ Y_1,\ \ldots\ldots,\ Y_7,\ Z), \qquad (3\text{-}3a)$$

where

$$L_a L^a = (T)^2-(X)^2-(Y_1)^2-\ \ldots\ldots\ -(Y_7)^2-(Z)^2 = 0,\ T > 0. \quad (3\text{-}3b)$$

The direction of L^a, from the above discussions, is determined uniquely by a point on S^+, and we denote this point by P (Fig. 1) ;

$$P(X/T,\ Y_1/T,\ \ldots\ldots,\ Y_2/T,\ Z/T) \in \boldsymbol{R}^9. \qquad (3\text{-}4a)$$

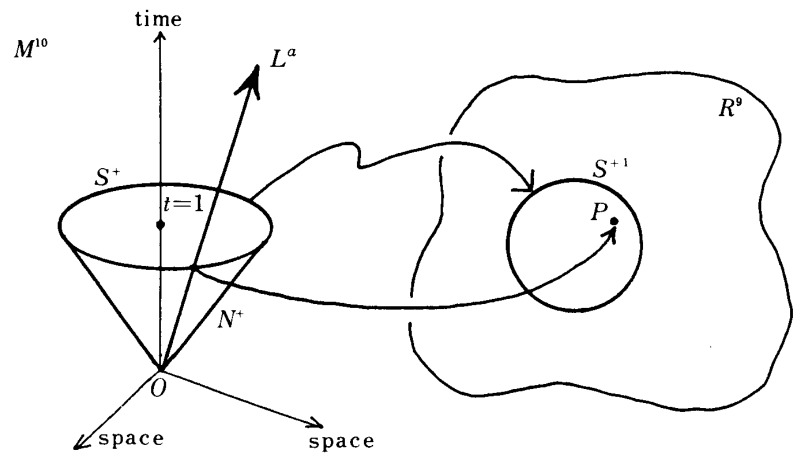

Fig. 1 $S^+ = S^8$ as the intersection of N^+ with a spatial hyperplane t = 1

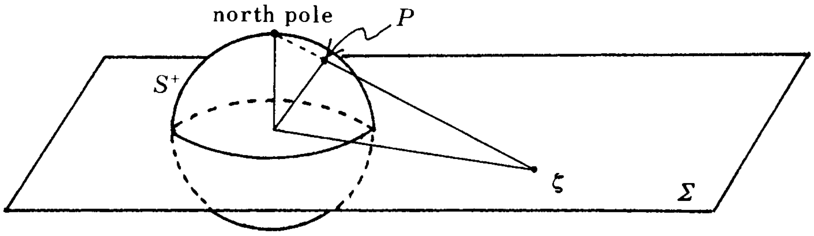

Fig. 2 The stereographic projection of S^+ to octonionic Gauss plane Σ

These nine tuples, of couse, satisfy the same condition as (3-2).

Let us consider an extended Riemann sphere on the octonionic Gauss plane $\boldsymbol{\Sigma}$ which is octonionic space including infinity. Since this extended Riemann sphere is equivalent to S^8, we may identify S^+ with this, and by the stereographic projection from the north pole $(0, \ldots\ldots, 0, 1) \in \boldsymbol{R}^9$ to $\overset{0}{z} = 0$, we may map any points on S^+ to the corresponding points labelled by octonions on $\boldsymbol{\Sigma}$. Now, we denote the mapping point on $\boldsymbol{\Sigma}$ from the point P of (3-4a) by $\boldsymbol{\zeta}$ (Fig. 2). Then by this $\boldsymbol{\zeta}$, P of (3-4a) may be rewritted as follows ;

$$P\left(\frac{\boldsymbol{\zeta}+\bar{\boldsymbol{\zeta}}}{N(\boldsymbol{\zeta})+1}, \frac{2\langle \boldsymbol{i}_1, \boldsymbol{\zeta}\rangle}{N(\boldsymbol{\zeta})+1}, \ldots\ldots, \frac{2\langle \boldsymbol{i}_7, \boldsymbol{\zeta}\rangle}{N(\boldsymbol{\zeta})+1}, 1-\frac{2}{N(\boldsymbol{\zeta})+1}\right), \tag{3-4b}$$

or coversely, by the components of P, $\boldsymbol{\zeta}$ may by written as follws ;

$$\boldsymbol{\zeta} = \frac{1}{1-(Z/T)}\left(\frac{X}{T}+\boldsymbol{i}_1\frac{Y_1}{T}+\ \ldots\ldots\ +\boldsymbol{i}_7\frac{Y_7}{T}\right). \tag{3-5}$$

Thus, we certainly establish one-to-one correspondence between a null direction in M^{10} and an octonion.

Although $\boldsymbol{\zeta}$ of (3-5) corresponds to any points on S^+, it is difficult to represent the north pole by it, since $\boldsymbol{\zeta}$ is infinity at this point. Therefore, in order to avoid using this infinite coodinate, we write $\boldsymbol{\zeta}$ by a pair $(\boldsymbol{\xi}, \boldsymbol{\eta})$ of octonions as follows ;

$$\boldsymbol{\zeta} = \boldsymbol{\xi}\boldsymbol{\eta}^{-1}, \tag{3-6}$$

and then the north pole may be represented by $\boldsymbol{\xi} = 1$ and $\boldsymbol{\eta} = 0$. These are to be projective octonionic coodinates, that is ; the pair $(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)$ are identified, if $\boldsymbol{\xi}\boldsymbol{\eta}^{-1} = \boldsymbol{\xi}^*\boldsymbol{\eta}^{*-1}$ or $\boldsymbol{\eta} = \boldsymbol{\eta}^* = 0$[7]. Thus, we may regard S^+ as a realization of an octonionic projective line $\boldsymbol{O}P^1$.

4. Vectorial Octonions and Spinorial Octonions

In sec. 3, we considered correspondance between a null line in M^{10} and an octonion. Although a null line certainly corresponds to an octonion, since the octonion includes an infinity, instead of it, we considered two octonions in the relation (3-6). In this section, we show, in the viewpoint of the octonionic transfomation law, the relations between these three octonions.

Let us cosider a vector $V_i (i = 1, 2, \ldots\ldots, 8)$ in the eight-dimensoinal Euclidian space E^8. We denote an $SO(8)$ transformation matrix by t_{ij}. Then V_i transforms under the $SO(8)$ transformation as follows ;

$$V_i \to t_{ij} V_j,\ t_{ik} t_{jk} = \delta_{ij},\ \det(t_{ij}) = 1. \tag{4-1}$$

We consider a rotation of the vector V_i in (l, m)-plane and denote the operator indicating this rotation by $t_{(lm)ij}$. Note that $t_{(lm)ij}$ is not only an element of $SO(8)$, but also an element of $SO(2)$. Then, a rotation in E^8 may be represented by composition of some 2-planes in E^8, that is,

$$t_{ij} = t_{(l_1 m_1) i k_1} t_{(l_2 m_2) k_1 k_2} \ldots\ldots t_{(l_n m_n) k_{n-1} j}, \tag{4-2}$$

where $n \leqq 28$ since $SO(8)$ is the 28-parameter group and $SO(2)$ is the 1-parameter group.

Let us take an octonionic representation of the vector V_1 as follows ;

$$\boldsymbol{V} = V_1 + \boldsymbol{i}_1 V_2 + \ldots\ldots + \boldsymbol{i}_7 V_8. \tag{4-3}$$

Now, we define following octonions

$$\boldsymbol{w}_{(lm)} = \cos(\theta/2) + \boldsymbol{i}_{m-1} \sin(\theta/2),\ (m \neq 1), \tag{4-4a}$$
$$\boldsymbol{w}_{(lm)} = \boldsymbol{i}_{l-1} \cos(\theta/2) - \boldsymbol{i}_{m-1} \sin(\theta/2),\ (l \neq m). \tag{4-4b}$$

Then we may see that the octonionic transformation for V of (4-3)

$$\boldsymbol{V} \to \bar{\boldsymbol{i}}_{l-1} (\boldsymbol{w}_{(lm)} \boldsymbol{V} \boldsymbol{w}_{(lm)}) \bar{\boldsymbol{i}}_{l-1},\ (\boldsymbol{i}_0 = 1) \tag{4-5}$$

corresponds to a rotation of V_1 in (l, m)-plane. Indeed, if we put

$$\begin{aligned} &\bar{\boldsymbol{i}}_{l-1} (\boldsymbol{w}_{(lm)} \boldsymbol{V} \boldsymbol{w}_{(lm)}) \bar{\boldsymbol{i}}_{l-1} \\ &= t_{(lm)1j} V_j + \boldsymbol{i}_1 t_{(lm)2j} V_j + \ldots\ldots + \boldsymbol{i}_7 t_{(lm)8j} V_j, \end{aligned} \tag{4-6}$$

then we obtain for the 8×8 matrix $t_{(lm)}$, dropping the indices i and j which indicate the (i, j)-component of the matrix,

$$t_{(lm)} = \begin{array}{c} \begin{array}{ccccccc} & l & & & m & & \end{array} \\ \left(\begin{array}{ccccccc} \mathbf{1}_p & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cos\theta & \cdots\ 0\ \cdots & -\sin\theta & & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & 0 & \mathbf{1}_q & 0 & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & \sin\theta & \cdots\ 0\ \cdots & \cos\theta & & 0 \\ 0 & \cdots & \cdots & 0 & & \mathbf{1}_r \end{array}\right) \end{array} \begin{array}{l} \\ \\ l \\ \\ \\ \\ m \\ \\ \end{array} \tag{4-7}$$

where $\mathbf{1}_p$, $\mathbf{1}_q$ and $\mathbf{1}_r$ are unit matrices of the rank p, q and r (where $p+q+r=6$), respectively, and this 8×8 matrix certainly expresses a rotation of a vector in (l, m)-plane. Thus, from (4-5) and (4-7), we obtain a general rotation of an octonion $\boldsymbol{V}$;

$$V \to \bar{\boldsymbol{i}}_{l_1-1}(\boldsymbol{w}_{(l_1m_1)}(\ldots\ldots(\bar{\boldsymbol{i}}_{l_n-1}(\boldsymbol{w}_{(l_nm_n)}V\boldsymbol{w}_{(l_nm_n)})\bar{\boldsymbol{i}}_{l_n-1})\ldots\ldots)\boldsymbol{w}_{(l_1m_1)})\bar{\boldsymbol{i}}_{l_1-1}. \quad (4\text{-}8)$$

We call octonions transforming as (4-8) *vectorial octonions*, and we denote a space of the vectorial octonions by $\boldsymbol{O}_V$.

To the vectorial octonions, we define *left- and right-acted spinorial octonions* as $\boldsymbol{O}_L$ and $\boldsymbol{O}_R$ respectively, of which elements transform, when the vectorial octonion transforms by (4-8), as follows ;

$$\boldsymbol{\phi} \to \bar{\boldsymbol{i}}_{l_1-1}(\boldsymbol{w}_{(l_1m_1)}(\ldots\ldots(\bar{\boldsymbol{i}}_{l_n-1}(\boldsymbol{w}_{(l_nm_n)}\boldsymbol{\phi}))\ldots\ldots) \text{ for } \boldsymbol{\phi} \in \boldsymbol{O}_L, \quad (4\text{-}9a)$$
$$\boldsymbol{\chi}' \to \pm((\ldots\ldots((\boldsymbol{\chi}'\boldsymbol{w}_{(l_nm_n)})\bar{\boldsymbol{i}}_{l_n-1})\ldots\ldots)\boldsymbol{w}_{(l_1m_1)})\bar{\boldsymbol{i}}_{l_1-1} \text{ for } \boldsymbol{\chi}' \in \boldsymbol{O}_R. \quad (4\text{-}10a)$$

If we write these octonions, $\boldsymbol{\phi}$ and $\boldsymbol{\chi}'$, as

$$\boldsymbol{\phi} = \phi_0 + \boldsymbol{i}_1\phi_1 + \ldots\ldots + \boldsymbol{i}_7\phi_7, \quad (4\text{-}11)$$
$$\boldsymbol{\chi}' = \chi_0 + \boldsymbol{i}_1\chi_1 + \ldots\ldots + \boldsymbol{i}_7\chi_7, \quad (4\text{-}12)$$

then we may rewrite (4-9a) and (4-10a) as

$$\boldsymbol{\phi} \to \pm(t_{0A}\phi_A + \boldsymbol{i}_1 t_{1A}\phi_A + \ldots\ldots + \boldsymbol{i}_7 t_{7A}\phi_A), \quad (4\text{-}9b)$$
$$\boldsymbol{\chi}' \to \pm(t_{0'A'}\chi_{A'} + \boldsymbol{i}_1 t_{1'A'}\chi_{A'} + \ldots\ldots + \boldsymbol{i}_7 t_{7'A'}\chi_{A'}), \quad (4\text{-}10b)$$

and then, we may prove for t_{AB} and $t_{A'B'}$ ($A, B = 0, 1, \ldots\ldots, 7$; $A', B' = 0', 1', \ldots\ldots, 7'$)

$$t_{AB}t_{BC} = \delta_{AB},\ \det(t_{AB}) = 1, \quad (4\text{-}9c)$$
$$t_{A'C'}t_{B'C'} = \delta_{A'B'},\ \det(t_{A'B'}) = 1. \quad (4\text{-}10c)$$

Thus, since both t_{AB} and $t_{A'B'}$ satisfy the $SO(8)$ conditions, the octonionic trnsformations (4-9a) and (4-10a) are two different octonionic representations of $SO(8)$ rotations from (4-8).

Here, we return to the three octonions of (3-6). *If we define $\boldsymbol{\zeta}$ as a vectorial octonion, then $\boldsymbol{\xi}$ and $\boldsymbol{\eta}^{-1}$ may be regarded as a left- and right-acted spinorial octonion, respectively,* since a product of these two octonions becomes a vectorial octonion. Therefore, if we put

$$\boldsymbol{\xi} = \boldsymbol{\phi} \in \boldsymbol{O}_L,\ \bar{\boldsymbol{\eta}} = \boldsymbol{\chi}' \in \boldsymbol{O}_{R'} \quad (4\text{-}13)$$

then we have

$$\boldsymbol{\zeta} = \boldsymbol{\phi}\boldsymbol{\chi}'/N(\boldsymbol{\chi}'), \quad (4\text{-}14)$$

and the projective octonionic line realization S^+ becomes an octonionic line in the two-dimensional octonionic space $\boldsymbol{O}_L \times \bar{\boldsymbol{O}}_R$.

5. Invariant Quantities of Vectorial and Spinorial Octonions and Principle of Triality

We have, for vectorial octonions and left- and right-acted spinorial octonions, the invariant quantities under the respective octonionic transformations (4-8), (4-9a) and (4-10a), respectively ;

$$\langle \overset{1}{\boldsymbol{V}}, \overset{2}{\boldsymbol{V}} \rangle = \text{inv., for } \overset{1}{\boldsymbol{V}}, \overset{2}{\boldsymbol{V}} \in \boldsymbol{O}_V, \tag{5-1}$$

$$\langle \overset{1}{\boldsymbol{\phi}}, \overset{2}{\boldsymbol{\phi}} \rangle = \text{inv., for } \overset{1}{\boldsymbol{\phi}}, \overset{2}{\boldsymbol{\phi}} \in \boldsymbol{O}_L, \tag{5-2}$$

$$\langle \overset{1}{\boldsymbol{\chi}}', \overset{2}{\boldsymbol{\chi}}' \rangle = \text{inv., for } \overset{1}{\boldsymbol{\chi}}', \overset{2}{\boldsymbol{\chi}}' \in \boldsymbol{O}_R, \tag{5-3}$$

Furthermore, we have

$$\langle \boldsymbol{V}, \boldsymbol{\phi\chi}' \rangle = \text{inv., for } \boldsymbol{V} \in \boldsymbol{O}_V, \boldsymbol{\phi} \in \boldsymbol{O}_L, \boldsymbol{\chi}' \in \boldsymbol{O}_R, \tag{5-4a}$$

since $\boldsymbol{\phi\chi}' \in \boldsymbol{O}_V$ from (4-9a), (4-10a) and (4-8), and (4-13a) hold for any two vectorial octonions.

Note that, from (2-7), (5-4a) may be also written as follows ;

$$\langle \boldsymbol{V}, \boldsymbol{\phi\chi}' \rangle = \langle \overline{\boldsymbol{\phi}\boldsymbol{\chi}'}, \overline{\boldsymbol{V}} \rangle = \langle \overline{\boldsymbol{\chi}}', \overline{\boldsymbol{V}}\boldsymbol{\phi} \rangle, \tag{5-4b}$$

and that $\boldsymbol{O}_V$, $\boldsymbol{O}_L$ and $\boldsymbol{O}_R$ three octonionic spaces which afford the three inequivalent octonionic representations of $SO(8)$. Then, we can find an interesting law which is called the *principle of triality*[8] ; *the three octonionic spaces, which are the vectorial octonionic space* $\boldsymbol{O}_V$*, the left-acted spinorial octonionic space* $\boldsymbol{O}_L$ *and the right-acted spinorial octonionic space* $\boldsymbol{O}_R$*, are all on an equal footing.*

6 . Conclusions

In the previous sections, we could see that octonions may be physically associated with a null cone in M^{10}. In consequence, a null line in M^{10} is equivalent to a projective octonionic line, which is an octonionic line in $\boldsymbol{O}_L \times \overline{\boldsymbol{O}}_R$, where $\boldsymbol{O}_L$ and $\boldsymbol{O}_R$ is the left- and right-acted spinorial octonionic space, respectively.

In the superstring theory, $SO(8)$-spinor play an important role. Mereover, in this theory, by using a light cone gauge, the important space-time quantites are $SO(8)$-tensorial objects. Our octonions may be corresponded to these quantities, that is, the vectorial octonions are octonionic representations, and the two spinorial octonions are octonionic representations, to $SO(8)$-vectors and two $SO(8)$-spinors, respetively.

However, the most advantage of our representations is that we can uniformly deal with a term of octonionns their objects. Furthermore, we may guess that the plinciple of triality suggests that our three octonions, which are equivalent by this plinciple, are some elementaly objects in the superspace descriptions.

Acknowredgments

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Appendix A : Useful Formulas of Octonions

We here give some formulas with respect to the associator of octonions[9].

Let us put $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in \boldsymbol{O}$. Then the associator (2-5) is pure imaginary, and we have

$$[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}] = [\boldsymbol{b},\ \boldsymbol{a},\ \boldsymbol{c}] = -[\boldsymbol{b},\ \boldsymbol{a},\ \boldsymbol{c}] = -[\bar{\boldsymbol{a}},\ \boldsymbol{b},\ \boldsymbol{c}], \tag{A1}$$

$$[\boldsymbol{a},\ \boldsymbol{ba},\ \boldsymbol{c}] = [\boldsymbol{a},\ \bar{\boldsymbol{a}}\boldsymbol{b},\ \boldsymbol{c}] = \boldsymbol{a}[\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}] = [\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}]\bar{\boldsymbol{a}}. \tag{A2}$$

From (A1), we may prove the important identities as follows ;

$$(\boldsymbol{ab})(\boldsymbol{ca}) = \boldsymbol{a}(\boldsymbol{bc})\boldsymbol{a}, \tag{A3}$$

where one must note that $(\boldsymbol{ba})(\boldsymbol{ac}) \neq \boldsymbol{ba}^2\boldsymbol{c}$.

Appendix B : $SO(8)$ Soldering Forms induced by Octonions

If we represent $\boldsymbol{V}$, $\boldsymbol{\phi}$ and $\boldsymbol{\chi}'$ as (4-3), (4-11) and (4-12), respectively, then (5-2) may be also written as follws ;

$$\langle \boldsymbol{V},\ \boldsymbol{\phi}\boldsymbol{\chi}' \rangle = \sigma_{iAA'} V_i \phi_A \chi_{A'}, \tag{B1}$$

where, denoting the 8×8 matrices $\sigma_{iAA'}$ by σ_i whose components are labeled as (A, A'), we have

$$\sigma_1 = \left(\begin{array}{cc|cc|cc|cc}
1 & & & & & & & \\
 & -1 & & & & & & \\
\hline
 & & -1 & & & & & \\
 & & & -1 & & & & \\
\hline
 & & & & -1 & & & \\
 & & & & & -1 & & \\
\hline
 & & & & & & -1 & \\
 & & & & & & & -1
\end{array}\right), \quad
\sigma_2 = \left(\begin{array}{cc|cc|cc|cc}
 & 1 & & & & & & \\
1 & & & & & & & \\
\hline
 & & & 1 & & & & \\
 & & -1 & & & & & \\
\hline
 & & & & & 1 & & \\
 & & & & -1 & & & \\
\hline
 & & & & & & & -1 \\
 & & & & & & 1 &
\end{array}\right),$$

$$\sigma_3 = \left(\begin{array}{cccc|cccc} & & 1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ \hline & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & -1 & & & \\ & & & & & -1 & & \end{array}\right), \quad \sigma_4 = \left(\begin{array}{cccc|cccc} & & & 1 & & & & \\ & & 1 & & & & & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & -1 & & & \end{array}\right),$$

$$\sigma_5 = \left(\begin{array}{cccc|cccc} & & & & 1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \\ \hline 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{array}\right), \quad \sigma_6 = \left(\begin{array}{cccc|cccc} & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & & -1 \\ & & & & & & 1 & \\ \hline & -1 & & & & & & \\ 1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \end{array}\right),$$

$$\sigma_7 = \left(\begin{array}{cccc|cccc} & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & 1 & & & \\ & & & & & -1 & & \\ \hline & & -1 & & & & & \\ & & & 1 & & & & \\ 1 & & & & & & & \\ & -1 & & & & & & \end{array}\right), \quad \sigma_8 = \left(\begin{array}{cccc|cccc} & & & & & & & 1 \\ & & & & & & -1 & \\ & & & & & 1 & & \\ & & & & 1 & & & \\ \hline & & & -1 & & & & \\ & & -1 & & & & & \\ & 1 & & & & & & \\ 1 & & & & & & & \end{array}\right). \tag{B2}$$

From (B2), we have for these $\sigma_{iAA'}$ identities

$$\sigma_{(i|AA'|}\sigma_{j)BA'} = \delta_{ij}\delta_{AB}, \tag{B3a}$$

$$\sigma_{(i|AA'|}\sigma_{j)AB'} = \delta_{ij}\delta_{A'B'}. \tag{B3b}$$

Moreover, from (B1), we have also

$$\sigma_{i(A|A'}\sigma_{i|B)B'} = \sigma_{iA(A'|}\sigma_{iB|B')} = \delta_{AB}\delta_{A'B'}. \tag{B4}$$

The identities (B3) and (B4) agree with the conditions for the Clebsch-Gordan coefficients or *the* $SO(8)$*-soldering forms* which couple the $SO(8)$-vector to the two $SO(8)$-reduced spinors.