

Superstring Theory, Triality and Octonions

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Summary

It has been shown, recently, that the division algebras are closely related to the underlying symmetries of the superstring theories or Kaluza-Klein theory of unification of all physical forces. Especially octonions play a specially important part in these theories implicitly in 10 dimensional superstring theories through "triality". While in the Kaluza-Klein theory, torsion plays an important part. These triality, parallelizing torsions and octonions in the theory are closely related, an explicit use of octonions in the theory is quite desirable such as an invariant action or a Lagrangian, Maurer-Cartan-Schouten differential equation but has not yet been done,

For the purpose, we have developed a theory of functions of an octonion variable which will be of use if the theory is explicitly expressed by an octonion form.

In this paper, we present an attempt to formulate these superstring theories in an explicit octonion form so that the structure of the theory will be more transparent. For this purpose, we look superstring theory of Green-Schwarz superspace theory from the view point of an octonion formulation of the theory.¹⁾

§ 1 . Introduction

In the superstring theory of Green-Schwarz, the theory is invariant under supersymmetric transformations iff the following conditions are satisfied :

$$\Gamma_{\mu}\Psi_{[1}\bar{\Psi}_2\Gamma^{\mu}\Psi_{3]} = 0 \quad (1.1)$$

(*) After the paper was submitted to the Editor we have come to know (Dr. T. Kayano have called attention to the paper by I. Oda, T. Kimura and A. Nakamura : Prog. Theor. Phys. 80 (1988) 367.) that I. Oda et al. have published a paper in which they have reported the study of superparticles using an octonionic formulation. The results of the study is quite useful to our study of the octonionic (or higher Cayley-Dickson hypercomplex numbers) formulation of the superstring theory.

It can be shown that these equations hold only in the following special dimensions²⁾ of the space-time i. e.,

D= 3 : spinors are a Majorana type,

D= 4 : spinors are a chiral or Majorana type,

D= 6 : spinors are chiral,

D= 10 : spinors are Majorana and Weyl type.

In these dimensions of space-time, as was shown earlier by several authors that spinors and vectors are closely related to the division algebras of R, C, H and O, respectively.

Evans³⁾ expressed (1. 1) into the Γ matrix form

$$\Gamma_{m\alpha} (\beta \Gamma^m_{\gamma\delta}) = 0 \quad (1. 2)$$

and the identities satisfied by these Γ matrices are the basis for the invariance of the supersymmetric transformations of the Lagrangian of the superstring theory of Green-Schwarz. They derived the relation between the above identities and the triality and the triality is due to the nature of the division algebras.

In this paper, we deal mainly with the action of Green-Schwarz in the superspace. The action can be split into two parts and the second part is identified as the Wess-Zumino term. When, the arbitrary constant coefficient of the second term is to be put equal to one, the Wess-Zumino term becomes identical with the torsion and the curvature of a group manifold becomes zero. The torsion is a parallelizing torsion and the theory becomes a free fermion theory.

As Cartan and Schouten have shown, the parallelizing torsion can be equated to that the Riemann curvature becomes zero. By adding antisymmetric torsion to the connection of S^3 or S^7 , the Riemann curvature becomes zero.

The results are due to the existence of quaternions and octonions underlying the relations. Several attempts by several authors^{4),5),6)} have been made to derive the parallelizing torsion directly from the octonionic formulation in the supergravity theory of 11 dimension. However, it seems to us that none of the attempts are satisfactory in view of a complete theory of the octonionic formulation. We describe this point in section 4. Now, we would like to make it clear that the relation of Green-Schwarz theory to octonions in the following sections.

§ 2 . Green-Schwarz theory in the superspace.

As is well known, the superstring theory cannot be quantized by using a covariant gauge in the superspace. As a result, a light-cone gauge is used. Starting from the generalized action integral of Nambu-Goto action as follows :

$$S_1 = - \frac{1}{2\pi} \int \sqrt{-h} h^{\alpha\beta} \pi_\alpha^\mu \pi_{\beta\mu} d\sigma d\tau \quad (2. 1)$$

where

$$\pi_\alpha^\mu = \partial_\alpha X^\mu - i \bar{\theta}^A \Gamma^{\mu A} \partial_\alpha \theta^A$$

S_1 is clearly invariant under a supersymmetric transformation. By adding further a term, S_2 :

$$S_2 = \frac{1}{\pi} \int d\sigma d\tau \left(-i\varepsilon^{\alpha\beta} \partial_\alpha x^\mu (\bar{\theta}^1 \Gamma_\mu \partial_\beta \theta^1 - \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2) \right. \\ \left. + \varepsilon^{\alpha\beta} \bar{\theta}^1 \Gamma^\mu \partial_\alpha \theta^1 \bar{\theta}^2 \Gamma_\mu \partial_\beta \theta^2 \right) \quad (2.2)$$

we examine the significance of these S_1 and S_2 in the following.

1) Superspace σ - model.

We consider N=1 supersymmetry⁷⁾ (heterotic case) .

We have 10 bosonic coordinates X^a and sixteen fermionic coordinates θ^a . The generators of the supersymmetric transformation satisfy the following relations :

$$[P_a, P_b] = 0, [P_a, Q_a] = 0, \\ \{Q_a, Q_\beta\} = -2i\Gamma_{a\beta}^a P_a \quad (2.3)$$

and satisfying the following normalization conditions are :

$$tr(P_a P_b) = \eta_{ab}, tr(P_a Q_\alpha Q_\beta) = -i\Gamma_{a\alpha\beta} \quad (2.4)$$

The elements of the supertranslation h :

$$h = e^{iX^a P_a - \theta^a Q_a} \quad (2.5)$$

Using (2.3), we have

$$h^{-1} \partial_i h = iV_i^a P_a - \partial_i \theta^a Q_a \quad (2.6)$$

where

$$V_i^a = \partial_i X^a + \partial_i \theta^a \Gamma_{a\beta}^a Q^\beta \quad (2.7)$$

then S_1 is given by

$$-\frac{1}{2\pi} \int d^2\xi \eta^{ij} tr(h^{-1} \partial_i h)(h^{-1} \partial_j h) \quad (2.8)$$

The second term S_2 is the Wess-Zumino term corresponding to S_1 of (2.8) :

$$W_z = -\frac{1}{\pi} \int d^3\xi \varepsilon^{ijk} tr(h^{-1} \partial_i h)(h^{-1} \partial_j h)(h^{-1} \partial_k h) \quad (2.9)$$

To clarify the significance of the Wess-Zumino term, let us consider the following action : $S_1 + a(WZ)$, $a = a$ constant⁸⁾.

When $a = 1$, the torsion parallelizes the manifold : the Riemann curvature R_{abcd} becomes zero as a result, the geometry of the σ -model has a renormalization group infrared fixed point at $a = 1$.

(i) Furthermore, the theory is an infinite dimensional local symmetry and it characterizes Kac-Moody algebra. In the case of Green-Schwarz, the action is given as $S_1 + S_2$. The superstring theory is κ symmetric. Due to the κ symmetry, the coefficient of Wess-Zumino term cannot be quantized. In the flat space, the existence of Wess-

Zumino term leads to a torsion due to the anticommutation relation $\{Q_\alpha, Q_\beta\} = -2i\Gamma_{\alpha\beta}^a P_a$.

As described later, the same holds in the case of octonions, the vector fields E_a and E_b made by octonions satisfy a commutation relation $[E_a, E_b](x) = -T^c{}_{ab}(x)E_c(x)$ as described in detail in the next section.

Now we deal with torsions.

(ii) Differential geometry in superspace.⁷⁾

Let $z^M = (x^m, \theta^\mu)$ be a point in the superspace, x^m be bosonic world coordinates, and θ^μ be anti-commuting fermionic world coordinates. Define a one-form $\{e^A\}$ as

$$e^A = dz^M e_M^A$$

We restrict ourselves to the case of $N = 1$, the case of ten dimension (heterotic) .

In a flat space, the superconnection is given by $\omega_A^B = 0$ and the non-vanishing components of torsion are $T_{a\beta}^a = 2\Gamma_{a\beta}^a$. The torsion is, by definition, given by

$$de^a = e^\alpha e^\beta \Gamma_{\alpha\beta}^a, \quad de^\alpha = 0 \quad (2.10)$$

The solution of these equations are given by

$$e^a = dx^a + d\theta^\alpha \Gamma_{\alpha\beta}^a \theta^\beta, \quad e^\alpha = d\theta^\alpha \quad (2.11)$$

and the vielbeins are given as

$$e_M^A = \begin{pmatrix} e_m^a = \delta_m^a & e_\beta^a = 0 \\ e_\mu^a = \delta_\mu^a \Gamma_{\alpha\beta}^a \theta^\beta, & e_\mu^\alpha = \delta_\mu^\alpha \end{pmatrix} \quad (2.12)$$

Now, noticing that the following three-form H :

$$H = \frac{1}{2} e^\alpha e^\beta e^\gamma \Gamma_{\alpha\beta\gamma}^a \quad (2.13)$$

is a closed form : $dH = 0$.

This equation can be shown, using the following identities for Γ matrices, as :

$$\Gamma_{\alpha\beta\gamma}^a \Gamma_{\gamma\delta}^a + \Gamma_{\alpha\beta\gamma}^a \Gamma_{\alpha\delta}^a + \Gamma_{\alpha\beta\gamma}^a \Gamma_{\beta\delta}^a = 0 \quad (2.14)$$

Furthermore, H is exact :

$$H = dB, \quad B = -\frac{1}{2} e^\alpha e^\beta \Gamma_{\alpha\beta}^a \theta^\beta.$$

Then, H , both closed and exact, can be regarded as a Wess-Zumino term.

Equation (2.14) , as mentioned in Introduction, is equivalent to the equations (1.1) and (1.2) and is important to our octonionic formulation of the supersymmetric theory. The meaning of which will be discussed in the following sections.

Using the vielbein :

$$V_i^a = \partial_i x^a + \phi_i \theta^\alpha \Gamma_{\alpha\beta}^a \theta^\beta = \partial_i z^M e_M^a \quad (2.15)$$

$$\partial_i \theta^\alpha = \partial_i Z^M e_M^\alpha = V_i^\alpha \tag{2.16}$$

the Green-Schwarz action can be written as

$$I = S_1 + S_2 = \frac{1}{\pi} \int d^2 \xi \left[\frac{1}{2} \eta^{ij} V_i^a V_j^b \eta_{ab} + i \epsilon^{ij} \partial_i Z^N \partial_j Z^M B_{MN} \right] \tag{2.17}$$

where we have used the relation $H = dB$.

§ 3 . Triality.

Evans³⁾ considers the following map to introduce the triality :

$$\gamma : V \times S_+ \times S_- \longrightarrow R, \quad (\nu, \xi, \eta) \longrightarrow \gamma_{ia\dot{a}} \nu_i \xi_a \eta_{\dot{a}} \tag{3.1}$$

Based on the Fierz equation (1.2) for the γ matrices, he defined the triality. The H introduced by (2.13) in the last section : $H = 1/2 (e^\alpha e^\beta e^\gamma) \Gamma_{\alpha\beta\gamma}$ when we consider $e^\alpha \in V, e^\alpha \in S_+, e^\beta \in S_-$, the equations (2.13) and (3.1) are the condition for the triality. This relation can be expressed, by using three octonions. Let A, B, C , where A be a vector octonion, B and C be spinor octonions, then an adequate octonion form be

$$H = \left(\bar{A} \cdot (BC) \right)$$

where $(A \cdot B)$ is a scalar product, then we have the following identities :

$$\left(\bar{A} \cdot (BC) \right) = \left(\bar{B} \cdot (CA) \right) = \left(\bar{C} \cdot (AB) \right).$$

From these relations, the nature of the triality of H becomes more transparent. Now, we look at the action of Green-Schwarz given in (2.17) . The first term is of the following form :

$$\bar{A} \cdot A = A \cdot \bar{A} = n(A)$$

While the second term is to be derived from the three form H and is of the following form :

$$\left(\bar{A} \cdot (BC) \right)$$

In this connection, we have to note that this triality is also related to the Jordan algebra $H_3(A)^{39)}$ as Gamba has described. To reveal a deeper meaning of the triality, we need a thorough study of the octonions.

We do not describe the relation between the triality and the outer automorphisms of $SO(8)(D_4)$ here⁹⁾.

§ 4 . Octonions

Hasiewicz and Lukierski¹⁰⁾ have given an extended equation of Cartan-Maurer equation for S^7 . They introduced an octonion one-form on S^7 :

$$\omega(x) = \bar{x} \cdot dx = -d\bar{x} \cdot x = -\bar{\omega}(x) = \omega^a(x)e_a \quad (4.1)$$

where e_a are octonion units and satisfy the following equations :

$$e_a e_b = -\delta_{ab} + f_{ab}^c e_c, \quad \bar{e}_a = -e_a, \quad (4.2)$$

The two-form is

$$\begin{aligned} d\omega(x) &= \frac{1}{2} d\bar{x} \wedge dx \\ &= \frac{1}{2} \omega(x) \wedge \omega(x) + \frac{1}{2} (\omega(x)\bar{x}, x, \omega(x)) \end{aligned} \quad (4.3)$$

where the last term in (4.3) is the associator of octonions and is

$$\left(\omega(x)\bar{x}, x, \omega(x) \right)^a = 2\Phi_{bc}^a(x) \omega^b(x) \wedge \omega^c(x) \quad (4.4)$$

$$d\omega^a(x) = T_{bc}^a(x) \omega^b(x) \wedge \omega^c(x) \quad (4.5)$$

here we have the following :

$$T_{bc}^a(x) = -f_{bc}^a + \Phi_{bc}^a(x). \quad (4.6)$$

The $T_{bc}^a(x)$ is the parallelizing torsion and agrees with that of Rooman's :

$$S^+ = -\alpha_{abc} - \frac{1}{2} \left\{ e_a [e_b, x, \overline{e_c x}] \right\}$$

The equality of S to $T_{bc}^a(x)$ can be derived if we put

$$x = (1 + \xi) (1 - \xi)^{-1}$$

in (4.1) .

Since Rooman obtained the torsion from the condition of parallelizability, the torsion given by (4.6) is also parallelizable.

Hasiewicz and Lukierski¹⁰⁾, starting from the commutation relation for the vector field E_a in the S^7 sphere :

$$[E_a, E_b](x) = -T_{ab}^c(x) E_c(x), \quad (a, b = 1, 2, \dots, 7) \quad (4.7)$$

and because the equations agree with eq. (4.5), defined $T_{bc}^a(x)$ as a parallelizing torsion.

§ 5 . Fierz Identity.

As pointed out by Evans and referred in this paper that the eq. (1.2) is the central core of the demonstration that the torsion H should be closed. We, in this section, present a different view of Goddard, Olive and Nahm¹¹⁾ to looking at the formulas (1.2).

Let us consider a group G' which contains a group G as a subgroup. Then G'/G is

a symmetric space and the generator of the tangent space transforms under the G as a fermion. Separate the generator g' , an element belonging to the g' group, into an even part and an odd part :

$$g' = g + p \quad (5.1)$$

The even generators T^i satisfy

$$[T^k, T^l] = if^{klm} T^m \quad (5.2)$$

Odd generators p^α are orthogonal to the even generators and transform, in a similar way to fermions, as follows :

$$[T^i, p^\alpha] = ip^\beta M_{\beta\alpha}^i \quad (5.3)$$

The symmetric space assumes the following relations to be valid for its elements : let p^α, T^i , satisfy

$$[p^\alpha, p^\beta] = X_l^{\alpha\beta} T^l \quad (5.4)$$

choosing

$$T_\gamma(T^i T^j) = y\delta_{ij}, \quad T_\gamma(p^\alpha p^\beta) = y\delta_{\alpha\beta}$$

we have

$$iyX_l^{\alpha\beta} = T_\gamma(T^l[p^\alpha, p^\beta]) = T_\gamma([T^l, p^\alpha], p^\beta) = iyM_{\alpha\beta}^l$$

and we obtain

$$[p^\alpha, p^\beta] = iM_{\alpha\beta}^l T^l$$

Thus, using (5.3) and (5.4), we obtain the relation

$$[[p^\alpha, p^\beta], p^\gamma] = \sum_{i=1}^{dimg} M_{\alpha\beta}^i M_{\gamma\delta}^i p^\delta$$

This relation shows that $\sum_{i=1}^{dimg} M_{\alpha\beta}^i M_{\gamma\delta}^i$ is the Riemann tensor of the symmetric space.

Then, by the Jacobi identity for p^α generators, we obtain the Fierz equation :

$$\sum_{i=1}^{dimg} \left(M_{\alpha\beta}^i M_{\gamma\delta}^i + M_{\beta\gamma}^i M_{\alpha\delta}^i + M_{\gamma\alpha}^i M_{\beta\delta}^i \right) = 0$$

These relations and octonions reveal a close relationship between them but it is not yet clear whether these relations can completely be reduced to the nature of octonions or not.

§ 6 . Extension of octonions to 16-nions.

As described above, superstring theory uses 10 dimensional spinors and vectors¹²⁾. Even though the central core of these quantities is octonions, if we deal with these

quantities in $(9, 1)$ space-time we have to extend octonions to 16-nions of Cayley-Dickson algebras whose quadratic signatures are

$$(+-----) \text{ or } (++++-----) \text{ or } (++++-----)$$

which are the space-time of $(15, 1)$, $(14, 2)$ or $(12, 4)$, respectively. Such a hypercomplex system can be given by complexifying the 16-nions and the theory of functions of such a hyperecomplex variable can be readily made using the theory we have already developed as given in ref.¹³⁾

However, we have to use vectors and spinors in the $(9, 1)$ space-time instead of 16-nions in the $(15, 1)$ space-time. In order that a spinors in the $(9, 1)$ space-time remains a spinor in the $(9, 1)$ space-time under a general rotation in the $(9, 1)$ space-time, the spinor must be a zero divisor lying on the light-cone whose vertex is at the origin of the coordinate in the $(15, 1)$ space-time as a hypercomplex number. The condition puts a severe constraint for the use of 16-nions for the light-cone spinors of $(9, 1)$ space-time.

The study on this problem is in progress and will be published in a near future.

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