Spinor Formalism to the Poincaré Gauge Theory II

— massless Lorentz gauge field in the case $C_1 = C_2 = C_3 = 0$ —

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Abstract

In a previous paper we have formulated a spinor formalism for Poincaré gauge theory. This formalism introduces the notion of a null tetrad in a remarkably natural way, and will be useful to study massless fields. In this paper we apply our formalism to the investigation of a "null field" in a free Lorentz gauge field under the condition $C_1 = C_2 = C_3 = 0$. We find two such modes with spin-parity 1^+ and 1^- : they can not coexist at the same time, however.

§ 1 . Introduction

In a previous paper[1),*] we have formulated a spinor formalism for Poincaré gauge theory (PGT) with the most general Lagrangian density proposed by Hayashi[2)]. In PGT the Lorentz gauge field $A_{km\mu}$ as well as the translational gauge field $c_k{}^\mu$ may be a propagating field with positive energy.

In this paper we apply our spinor formalism to the investigation of a "null field"[3),4),5)] (which is a typical massless mode) in a free Lorentz gauge field under the condition $C_1 = C_2 = C_3 = 0$. The spinor technique has grown up with the study of general relativity, and is especially successful when applied to the study of gravitational radiation. Accordingly, our spinor formalism could have an advantage over the tensor formalism to investigate the propagating massless gauge fields.

In **§ 2** under the weak field approximation, we give the equations for the free Lorentz gauge field $A_{km\mu}$, and in **§ 3** their spinor form. **§ 4** and **§ 5** are devoted to preparations for rewriting the equations obtained in a previous section in a form written in terms of dyad components. In **§ 6** we investigate the existence of a "null field" in PGT, and find such a field that has spin-parity 1^- or 1^+ under the adequate parameter conditions. The final section is devoted to conclusions.

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* We shall refer this reference as I .

§ 2 . Weak field approximation

In this section we consider the weak field approximation for the Lorentz gauge field $A_{km\mu}$ in vacuum, i.e., $q = 0$. In order to do this, we assume that all the constants $a_i(i = 1\sim 6)$ are of the same order of magnitude, and normalize them as

$$a_i = -(1/\varkappa)b_i. \ (i = 1\sim 6) \tag{2.1}$$

Here $\varkappa$ is dimensionless Einstein's gravitational constant, i.e., $\varkappa = M_p{}^2 \times$ (Einstein's gravitational constant) $= 1.48\times 10^{-37}$.

For brevity, we assume here that $b_i = 0[1]$, and the field $b_k{}^\mu$ and A_{kmn} can be expanded into power series of $\varkappa^{1/2}$ in a weak field approximation. And neglecting the higher order terms, we put as follows :

$$b_k{}^\mu = \eta_k{}^\mu \tag{2.2}$$

and

$$A_{kmn} = \varkappa^{1/2}\overset{(1)}{A}_{kmn}. \tag{2.3}$$

Inserting these expansions into (I : 2.2.7) and (I : 2.3.1), and neglecting the higher order terms, we obtain the following equations :

$$\overset{(1)}{H}{}^{kmnp}{}_{,p} = 0 \tag{2.4}$$

and

$$\overset{(1)}{F}{}^{+kmnp}{}_{,p} = 0. \tag{2.5}$$

Here note that

$$F^{kmnp} = \varkappa^{1/2}\overset{(1)}{F}{}^{kmnp} + \varkappa\overset{(2)}{F}{}^{kmnp} + \ \ldots\ldots \tag{2.6}$$

with

$$\overset{(1)}{F}_{kmnp} = 2\overset{(1)}{A}_{km[p,n]} \tag{2.7}$$

and*

$$\begin{aligned} J^{kmnp} &= (-1/\varkappa)H^{kmnp} \\ &= (-1/\varkappa)\{\varkappa^{1/2}\overset{(1)}{H}{}^{kmnp} + \varkappa\overset{(2)}{H}{}^{kmnp} + \ \ldots\ldots\}. \end{aligned} \tag{2.8}$$

(H^{kmnp} is the same as J^{kmnp} when b_i is inserted into it in place of a_i.)

Finally, let us consider the Poincaré gauge transformation in the weak field approximation. The general transformation that leaves the fields weak is of the form

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* J^{kmnp} corresponds to H^{kmnp} of I .

$$x^\mu \to x'^\mu = x^\mu + \overset{(0)}{\omega}{}^\mu{}_\nu x^\nu + \varepsilon^\mu$$
$$q(x) \to q'(x') = [1 + (i/2)(\overset{(0)}{\omega}_{km} + \varkappa^{1/2}\overset{(1)}{\omega}_{km}(x))S^{km}]q(x) \tag{2.9}$$

where $\overset{(0)}{\omega}{}^\mu{}_\nu$ $(= \overset{(0)}{\omega}_{km}\eta^{k\mu}\delta^m{}_\nu)$ and ε^μ are arbitrary infinitesimal constants. Under this transformation, the field $\overset{(1)}{A}_{kmn}$ transforms as

$$\begin{aligned}\overset{(1)}{A}_{kmn}{}' = \overset{(1)}{A}_{kmn} - \overset{(0)}{\omega}{}^l{}_k\overset{(1)}{A}_{lmn} - \overset{(0)}{\omega}{}^l{}_m\overset{(1)}{A}_{kln}\\ - \overset{(0)}{\omega}{}^l{}_n\overset{(1)}{A}_{kml} - \overset{(1)}{\omega}_{km,n}.\end{aligned} \tag{2.10}$$

§ 3. Field equations and gauge transformations in a spinor form

Neglecting the super-script (0) and (1) in our approximation, we can immediately obtain from (I : 4.1.12), (I : 4.1.13), (I : 4.2.2) and (I : 4.2.3) the spinor equations which are equivalent to the field equations (2.4) and (2.5). And we can, in a similar way, get from (2.7) the corresponding three spinor equations.

First of all, we see that eq. (2.4) is equivalent to the following two spinor equations :

$$\begin{aligned}4b_4\nabla^{\dot{A}}{}_H\Phi^{BH} + \nabla_{\dot{G}H}\{(2b_3 - b_5)X^{\dot{A}\dot{G}BH} - (2b_3 + b_5)X^{BH\dot{A}\dot{G}}\}\\ + 6\nabla^{\dot{A}B}\{(b_1 + 6b_6)\Lambda - (b_1 - 6b_2)\Lambda^*\} = 0\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}3b_2\nabla^{\dot{A}}{}_H\Psi^{BCDH} + 2b_4\nabla^{\dot{A}(B}\Phi^{CD)}\\ - \nabla_{\dot{G}}{}^{(B}\{(2b_3 - b_5)X^{\dot{A}\dot{G}CD)} - (2b_3 + b_5)X^{CD)\dot{A}\dot{G}}\} = 0,\end{aligned} \tag{3.2}$$

where $\nabla_{\dot{A}B} = \sigma^k{}_{\dot{A}B}\partial_k$.

Next, we find out from (2.5) the following two identities :

$$\nabla_{\dot{A}}{}^F\Psi_{BCDF} - \nabla^{\dot{G}}{}_{(B}X_{CD)\dot{A}\dot{G}} - \nabla_{\dot{A}(B}\Phi_{CD)} = 0 \tag{3.3}$$

and

$$\nabla^{\dot{G}H}X_{BH\dot{A}\dot{G}} - 2\nabla_{\dot{A}}{}^H\Phi_{BH} + 3\nabla_{\dot{A}B}\Lambda = 0. \tag{3.4}$$

And also, from (2.7) we get the following four spinor equations :

$$\nabla_{\dot{E}(A}\underset{(A)}{\psi}{}^{\dot{E}}{}_{BCD)} = \Psi_{ABCD}, \tag{3.5}$$

$$-\nabla_{\dot{E}F}\underset{(A)}{\psi}{}^{\dot{E}F}{}_{AB} + 4\nabla^{\dot{E}}{}_{(A}\underset{(A)}{\varphi}{}_{\dot{E}B)} = 4\Phi_{AB}, \tag{3.6}$$

$$\nabla^{\dot{E}}{}_{(B}\underset{(A)}{\psi}{}_{D)\dot{A}\dot{C}\dot{E}} + \nabla_{\dot{A}(B}\underset{(A)}{\varphi}{}_{D)\dot{C}} + \nabla_{\dot{C}(B}\underset{(A)}{\varphi}{}_{D)\dot{A}} = -X_{\dot{A}\dot{C}BD} \tag{3.7}$$

and

$$\underset{(A)}{\varphi}{}^{\dot{E}F}{}_{,\dot{E}F} = 2\Lambda, \tag{3.8}$$

where $\underset{(A)}{\psi}{}_{\dot{E}ABC}$ and $\underset{(A)}{\varphi}{}_{\dot{E}A}$ are irreducible spinors of the spinor equivalent to the Lorentz gauge field $\overset{(1)}{A}_{kmn}$, defined just like $\psi_{\dot{E}ABC}$ and $\varphi_{\dot{E}A}$ of $\mathscr{B}_{\dot{A}B\dot{C}D\dot{E}F}$.

Finally, let us consider the gauge transformation (2.10). This can be rewritten in spinor form as follows :

$$\underset{(A)}{\tilde{\varphi}}{}_{\dot{E}A} = \underset{(A)}{\varphi}{}_{\dot{E}A} - (1/3)\nabla_{\dot{E}F}\lambda^{F}{}_{A} \tag{3.9}$$

and

$$\underset{(A)}{\tilde{\psi}}{}_{\dot{E}ABC} = \underset{(A)}{\psi}{}_{\dot{E}ABC} - \nabla_{\dot{E}(A}\lambda_{BC)}, \tag{3.10}$$

where λ_{AB} is the irreducible spinor of the spinor equivalent to ω_{km}.

§ 4 . Dyad formalism

In actually performing calculations it is convenient to introduce in spinor space S_2 a basis $\zeta_a{}^A = (\zeta_0{}^A,\ \zeta_1{}^A) = (o^A,\ \iota^A)$ (called a "dyad") satisfying the normalization conditions

$$\zeta_{aA}\zeta_b{}^A = \varepsilon_{ab},\ \zeta_{aA}\zeta^a{}_B = \varepsilon_{AB}, \tag{4.1}$$

where ε_{ab} plays the same role for the dyad components of spinors as ε_{AB} does for spinors.

In our approximation considered in section 2 we use the following concrete representations for a dyad $\zeta_a{}^A$, and $\sigma^\mu{}_{\dot{A}B}$:

$$\zeta_a{}^A = \delta_a{}^A,\ \text{i.e., } o^A = (1,\ 0),\ \iota^A = (0,\ 1) \tag{4.2}$$

and

$$\sigma^0{}_{\dot{A}B} = 2^{-1/2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad \sigma^1{}_{\dot{A}B} = 2^{-1/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2{}_{\dot{A}B} = 2^{-1/2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad \sigma^3{}_{\dot{A}B} = 2^{-1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

Then the spinors $\underset{(A)}{\varphi}{}_{\dot{A}B}$, $\underset{(A)}{\psi}{}_{\dot{A}BCD}$, Ψ_{ABCD}, Φ_{AB}, $X_{\dot{A}\dot{B}CD}$ are represented in terms of their dyad components $\underset{(A)}{\varphi}{}_m$, $\underset{(A)}{\psi}{}_M$, Ψ_m, Φ_M, X_M :

$$\underset{(A)}{\varphi}{}_{\dot{0}0} = \underset{(A)}{\varphi}{}_0,\ \underset{(A)}{\varphi}{}_{\dot{0}1} = \underset{(A)}{\varphi}{}_1,\ \underset{(A)}{\varphi}{}_{\dot{1}0} = \underset{(A)}{\varphi}{}_2,\ \underset{(A)}{\varphi}{}_{\dot{1}1} = \underset{(A)}{\varphi}{}_3,$$
$$\underset{(A)}{\psi}{}_{\dot{0}000} = \underset{(A)}{\psi}{}_0,\ \underset{(A)}{\psi}{}_{\dot{0}001} = \underset{(A)}{\psi}{}_1,\ \underset{(A)}{\psi}{}_{\dot{0}011} = \underset{(A)}{\psi}{}_2,\ \underset{(A)}{\psi}{}_{\dot{0}111} = \underset{(A)}{\psi}{}_3,$$
$$\underset{(A)}{\psi}{}_{\dot{1}000} = \underset{(A)}{\psi}{}_4,\ \underset{(A)}{\psi}{}_{\dot{1}100} = \underset{(A)}{\psi}{}_5,\ \underset{(A)}{\psi}{}_{\dot{1}110} = \underset{(A)}{\psi}{}_6,\ \underset{(A)}{\psi}{}_{\dot{1}111} = \underset{(A)}{\psi}{}_7,$$
$$\Phi_{00} = \Phi_0,\ \Phi_{01} = \Phi_1,\ \Phi_{11} = \Phi_2,$$
$$\Psi_{0000} = \Psi_0,\ \Psi_{0001} = \Psi_1,\ \Psi_{0011} = \Psi_2,\ \Psi_{0111} = \Psi_3,\ \Psi_{1111} = \Psi_4,$$
$$X_{\dot{0}\dot{0}00} = X_0,\ X_{\dot{0}\dot{0}01} = X_1,\ X_{\dot{0}\dot{0}11} = X_2,\ X_{\dot{0}\dot{1}00} = X_3,\ X_{\dot{0}\dot{1}01} = X_4,$$
$$X_{\dot{0}\dot{1}11} = X_5,\ X_{\dot{1}\dot{1}00} = X_6,\ X_{\dot{1}\dot{1}01} = X_7,\ X_{\dot{1}\dot{1}11} = X_8\ ;$$

,and also $\nabla_{\dot{A}B}$ in terms of

$$\begin{aligned}\nabla_{0\dot{0}} &\equiv \quad D = 2^{-1/2}(\partial_0+\partial_3),\\ \nabla_{1\dot{1}} &\equiv \quad \Delta = 2^{-1/2}(\partial_0-\partial_3),\\ \nabla_{0\dot{1}} &\equiv \quad \delta^* = 2^{-1/2}(\partial_1-i\partial_2),\\ \nabla_{1\dot{0}} &\equiv \quad \delta = 2^{-1/2}(\partial_1+i\partial_2).\end{aligned} \tag{4.4}$$

All the spinor equations obtained in a previous section are, using these representations, at once rewritten in terms of the dyad components. Then taking into account the invariance of the system of equations under SL(2, C)-transformations, we find that their equations are resolved into three parts, one of which is invariant and the remaining two are related to each other by a g_τ-transformation (which is defined in the next section).

§ 5. Change of spin frame*

Let us consider the SL(2, C)-transformations of a dyad $\zeta_a{}^A = (o^A, \iota^A)$:

$$\zeta_a{}^A = g_{ab}\zeta'_b{}^A \tag{5.1}$$

or

$$\begin{pmatrix} o^A \\ \iota^A \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} o'^A \\ \iota'^A \end{pmatrix},$$

where a, b, c and d are complex numbers satisfying $\det(g_{ab}) = ad-bc = 1$.

It is well-known that any elements of SL(2, C) group can be factorized as a product of the three typical elements

$$g_1(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},\ g_2(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix},\ g_3(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

where z is a complex parameter.

$g_1(z)$, $g_2(z)$ and $g_3(z)$ have the following interpretation :

$g_1(z)$……Null rotation around ℓ^μ,

$g_2(z)$……Boost in the ℓ^μ-n^μ plane and spatial rotation in the m^μ-$m^{*\mu}$ plane,

$g_3(z)$……Null rotation around n^μ,

where ℓ^μ, n^μ, m^μ and $m^{*\mu}$ are members of a null-tetrad $z_m{}^\mu$. In particular, $g_2(\exp.[i\phi/2])$-transformation may be better understood as a spatial ϕ-rotation in terms of a pair of real, orthogonal unit space-like vectors a^μ and b^μ definded from m^μ and $m^{*\mu}$ by

$$\begin{aligned}a^\mu &= 2^{-1/2}(m^\mu + m^{*\mu}),\\ b^\mu &= -i2^{-1/2}(m^\mu - m^{*\mu}).\end{aligned} \tag{5.2}$$

In fact, when the representations (4.2) and (4.3) are adopted, a^μ and b^μ are represented by $a^\mu = (0, 1, 0, 0)$ and $b^\mu = (0, 0, 1, 0)$, so that the above transformation is interpreted

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* Throughout this paper we use the same notations as those of Ref.5).

as a ϕ-rotation about x^3-axis.

By making use of the typical transformations $g_1(z) \sim g_3(z)$, we also compose for later use an important transformation g_τ :

$$g_\tau = g_3(1)g_1(-1)g_3(1) \tag{5.3}$$

by which each components of a null-tetrad $z_m{}^\mu$ transform like

$$\ell^\mu \to n^\mu,\ m^\mu \to -m^{*\mu},\ n^\mu \to \ell^\mu$$

By using the representations (4.2) and (4.3) for the above expressions, we find that this transformation corresponds to π-rotation in the x^1-x^3 plane.

It is covenient to note that the dyad components $\underset{(A)}{\varphi}{}_m$, $\underset{(A)}{\psi}{}_M$, Ψ_m, Φ_M, X_M and the operators D, Δ, δ, δ^* transform under the g_τ-transformation as

$$\begin{aligned}
&\varphi_0 \rightleftarrows \varphi_3,\ \varphi_1 \rightleftarrows -\varphi_2\ ;\ \psi_0 \rightleftarrows \psi_7,\ \psi_1 \rightleftarrows -\psi_6,\\
&\psi_2 \rightleftarrows \psi_5,\ \psi_3 \rightleftarrows -\psi_4\ ;\ \Psi_0 \rightleftarrows \Psi_4,\ \Psi_1 \rightleftarrows -\Psi_3,\\
&\Psi_2 \rightleftarrows \Psi_2\ ;\ \Phi_0 \rightleftarrows \Phi_2,\ \Phi_1 \rightleftarrows -\Phi_1\ ;\ X_0 \rightleftarrows X_8,\\
&X_1 \rightleftarrows -X_7,\ X_2 \rightleftarrows X_6,\ X_3 \rightleftarrows -X_5,\ X_4 \rightleftarrows X_4,
\end{aligned}$$

and

$$D \rightleftarrows \Delta,\ \delta \rightleftarrows -\delta^*,$$

where the subscribt (A) has been omitted.

Finally, we consider the space inversion transformation. This transformation is definded by

$$p(\zeta_a{}^A) = \zeta_{\dot{b}}{}^{\dot{A}} P^{\dot{b}}{}_{\dot{a}}, \tag{5.4}$$

with

$$P^{\dot{a}}{}_{\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where p is a projection operator by which any spinor in spin space S_2 is projected into the conjugate space $S_2{}^*$.

Noticing that the basis $\xi_{m\dot{A}B}$ transforms under this trasformation as

$$p(\xi_{m\dot{A}B}) = \xi_{n\dot{B}A} S_{nm}, \tag{5.5}$$

we can obtain the transformation property of a null-tetrad $z_m{}^\mu$, i.e.,

$$p(z_m{}^\mu) = S_{mn} z_n{}^\mu, \tag{5.6}$$

where

$$S_{mn} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

And also, we see that the spinor $\underset{(A)}{\varphi}{}_{\dot{A}B}$ transforms like

$$p(\underset{(A)}{\varphi}{}_{\dot{A}B}) = p(\underset{(A)}{\varphi}{}_m \xi_{m\dot{A}B}) = \underset{(A)}{\varphi}{}_m{}^* p(\xi_{m\dot{A}B})$$
$$= \underset{(A)}{\varphi}{}'_{A\dot{B}} = \underset{(A)}{\varphi}{}'_m \xi_{A\dot{B}},$$

and therefore

$$\underset{(A)}{\varphi}{}'_m = (\underset{(A)}{\varphi}{}_3{}^*, -\underset{(A)}{\varphi}{}_2{}^*, -\underset{(A)}{\varphi}{}_1{}^*, \underset{(A)}{\varphi}{}_0{}^*). \tag{5.7}$$

In the same way, we obtain for the spinor $\underset{(A)}{\psi}{}_{\dot{A}BCD}$

$$\underset{(A)}{\psi}{}'_M = (\underset{(A)}{\psi}{}_7{}^*, -\underset{(A)}{\psi}{}_6{}^*, \underset{(A)}{\psi}{}_5{}^*, -\underset{(A)}{\psi}{}_4{}^*, -\underset{(A)}{\psi}{}_3{}^*, \underset{(A)}{\psi}{}_2{}^*, -\underset{(A)}{\psi}{}_1{}^*, \underset{(A)}{\psi}{}_0{}^*). \tag{5.8}$$

6. Search for a "null field" in PGT

In this section we search for the possibility of existence of a "null field" in PGT. To do so, we investigate the plane wave solutions which may give the typical ones of such a null field. Consequently, we start with the assumption that all the fields depend only on one spatial coordinate x^3 and time x^0.

First of all, we consider the equations (3.1) ~ (3.4). It is easy to see that they can be rewritten in terms of the dyad components Ψ_m, Φ_M, X_M, Λ and their complex conjugates as follows :

$$g_1 D\Phi_2 = g_3 \Delta\Phi_0{}^* \tag{6.1a}$$
$$g_3 D\Phi_2 = g_1 \Delta\Phi_0{}^* \tag{6.1b}$$
$$g_1 D\Psi_3 = g_2 \Delta\Psi_1{}^* \tag{6.2a}$$
$$g_2 D\Psi_3 = g_1 \Delta\Psi_1{}^* \tag{6.2b}$$
$$D\{2(g_1\Phi_1 + g_3\Phi_1{}^*) + 3(g_1\Psi_2 + g_2\Psi_2{}^*)\} = 0 \tag{6.3a}$$
$$\Delta\{2(g_1\Phi_1 + g_3\Phi_1{}^*) - 3(g_1\Psi_2 + g_2\Psi_2{}^*)\} = 0 \tag{6.3b}$$
$$D(2\Phi_1 + 3\Psi_2 - 2X_4{}^*) = -\Delta X_0{}^* \tag{6.4a}$$
$$\Delta(2\Phi_1 - 3\Psi_2 + 2X_4{}^*) = DX_8{}^* \tag{6.4b}$$
$$D(g_1 X_6{}^* + g_2 X_2) = 0 \tag{6.5a}$$
$$\Delta(g_2 X_6{}^* + g_1 X_2) = 0 \tag{6.5b}$$
$$DX_2{}^* = -\Delta\Psi_0 \tag{6.6a}$$
$$\Delta X_6{}^* = -D\Psi_4 \tag{6.6b}$$
$$X_1 = \Phi_0{}^* + \Psi_1{}^* \tag{6.7a}$$
$$g_1 X_3 = -g_3\Phi_0 - g_2\Psi_1 \tag{6.7b}$$
$$g_1 X_5 = g_3\Phi_2 - g_2\Psi_3 \tag{6.8a}$$
$$X_7{}^* = -\Phi_2 + \Psi_3 \tag{6.8b}$$
$$g_4(X_4 - X_4{}^*) - (g_1 - g_2 - g_4)(\Psi_2 - \Psi_2{}^*) = 0 \tag{6.9a}$$
$$g_5(X_4 + X_4{}^*) + (g_1 + g_2 - g_5)(\Psi_2 + \Psi_2{}^*) = 0 \tag{6.9b}$$
$$\Lambda = \Psi_2 - X_4{}^*, \tag{6.10}$$

where we put

$$
\begin{aligned}
g_1 &= 2(2b_3 - b_5),\\
g_2 &= -2(3b_2 + 2b_3 + b5),\\
g_3 &= -2(2b_3 + 2b_4 + b_5),\\
g_4 &= 8(b_1 + b_3),\\
g_5 &= -4(b_5 + 12b_6).
\end{aligned}
$$

From these equations we find at once the d'Alembert equations :

$$
\begin{aligned}
(g_1^2 - g_3^2)\,\Box\,\Phi_M &= 0,\ (M = 0,\ 1,\ 2)\\
(g_1^2 - g_2^2)\,\Box\,\Psi_m &= 0,\ (m = 1,\ 2,\ 3)\\
(g_1^2 - g_2^2)\,\Box\,X_M &= 0.\ (M = (2,\ 6)
\end{aligned}
$$

From them we see that in the case where $g_1^2 \neq g_2^2$, $g_1^2 \neq g_3^2$, $g_4 \neq 0$ and $g_5 \neq 0$, $\Phi_M(M = 0, 1, 2)$, Ψ_1, Ψ_2, Ψ_3, X_2 and X_6 are all independent and satisfy the d'Alembert equation. However, Ψ_0, Ψ_4, X_0 and X_8 do not satisfy the d'Alembert equation even in that case. This means that the spinors Ψ_{ABCD} and $X_{\dot{A}\dot{B}CD}$ themselves can not represent any physical fields, but only the bispinor Φ_{AB} may represent a massless field with spin 1, These facts force us to put $g_1^2 = g_2^2$ $(\neq g_3^2)$. Because the spinors Ψ_{ABCD} and $X_{\dot{A}\dot{B}CD}$ could then become merely supplementary ones, i.e., some of their components may depend upon the components of Φ_{AB}, and others may be indifinite or zero. In the meantime, we find also from the equations under cosideration that Φ_0 and Φ_2 satisfy

$$\Delta\Phi_0 = 0,\ D\Phi_2 = 0. \qquad (6.1\text{ab})'$$

These equations show that Φ_0 is an arbitrary function of $x^0 + x^3$, and that Φ_2 is that of $x^0 - x^3$. On the other hand, we see also that Φ_1 must be a definite function of both $x^0 + x^3$ and $x^0 - x^3$ if it does not vanish. However, such a field stands in our way to our purpose. Fortunately we easily find that Φ_1 can indeed be eliminated* if we put either [I] $g_1 = -g_2$ and $g_4 = 0$, or [II] $g_1 = g_2$ and $g_5 = 0$. From now on, we study exclusively these two cases. Before going forward, we remark here the following two propositions :

(1) The equations under consideration are invariant under the g_T-transformation, so that it is enough to consider only a plane wave solution propagating in the positive direction along the x^3 axis.

(2) In both cases [I] and [II] , the field equations under consideration are invariant under the (extra) gauge transformations :

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\begin{aligned}
\tilde{X}_0 &= X_0 + F(x^0 + x^3), & \tilde{X}_1 &= X_1 + L^*(x^0 + x^3),\\
g_1\tilde{X}_3 &= g_1X_3 - g_2L(x^0 + x^3), & g_1\tilde{X}_5 &= g_1X_5 - g_2M(x^0 - x^3)
\end{aligned}
$$

* Since we are considering the special case of plane wave solutions, constant fields are excluded.

$$\tilde{X}_7 = X_7 + M^*(x^0 - x^3), \qquad \tilde{X}_8 = X_8 + H(x^0 - x^3),$$
$$\tilde{\Psi}_0 = \Psi_0 + G(x^0 + x^3), \qquad \tilde{\Psi}_1 = \Psi_1 + L(x^0 + x^3),$$
$$\tilde{\Psi}_3 = \Psi_3 + M(x^0 - x^3), \qquad \tilde{\Psi}_4 = \Psi_4 + K(x^0 - x^3),$$

with others being kept invariant, where (F, G, L) and (K, H,M) are arbitrary functions of $x^0 + x^3$ and $x^0 - x^3$, respectively.

We shall, by the proposition (1), assume that all the fields are functions of $x^0 - x^3$ only, and then, on account of (2), choose the functions H and K so that the fields $\tilde{X}_8$ and $\tilde{\Psi}_4$ will vanish. Then we have at once the following result in each case of [I] and [II] : (neglecting constant fields)

"Φ_2 and any one of Ψ_3, X_5 and X_7 are only independent and arbitrary functions of $x^0 - x^3$, and all the others are zero." (6.11)

This gives the following ones for the dyad components of energy-momentum tensor (I : 4.1.10) of Lorentz gauge field : *

"$\underset{(F)}{\chi}_8 \neq 0$, and all the other components are zero"

This result shows that the field represented by (6.11) is "null"[5]. The energy density of the null field is given by

$$\underset{(F)}{T}^{00} = (1/2)\underset{(F)}{\chi}_8 = -2[(g_1{}^2 - g_3{}^2)/g_1]\Phi_2{}^*\Phi_2 \tag{6.12}$$

no matter which one of Ψ_3, X_5 and X_7 we choose as an independent field. Here it should be remarked that the energy density (6.12) does not contain explicitly any one of Ψ_3, X_5 and X_7, and therefore one independent field of them is non-physical. It may be natural to remove such a non-physical field from the theory. Fortunately, we can carry it out by the extra gauge transformations (2). We choose an arbitrary function M such that the field X_7 is eliminated. Then, only the field Φ_2 becomes independent, and Ψ_3 and X_5 are given in terms of Φ_2 by

$$\Psi_3 = \Phi_2 \text{ and } X_5 = -[(g_2 - g_3)/g_1]\Phi_2. \tag{6.13}$$

Incidentally, a plane wave solution propagating in the negative direction of the x^3 axis can be obtained at once by the g_T-transformation of all the above results. And it is easy to verify that we can take them together with the above results as a general solution of the field equations under consideration.

We are now in a position to consider the equations (3.5) ~(3.8), together with (3.9) and (3.10). It should be noticed that the field equations to be satisfied by A_{kmn} or $\underset{(A)}{\varphi}_m$ and $\underset{(A)}{\psi}_M$ are originally of second order, so that if and only if the plane waves

* If we start with the assumption that all the fields are functions of $x^0 + x^3$ only, we have the result that

"$\underset{(F)}{\chi}_0 \neq 0$, and all the other components are zero."

propagating in the right and left directions along the x^3 axis are represented by the same "potential" function (generally, a linear combination of A_{kmn} or $\underset{(A)}{\varphi}_m$ and $\underset{(A)}{\psi}_M$), the potential function will satisfy the d'Alembert equation. Accordingly, we must consider here a general case where there are right- and left-moving plane waves, and assume that the fields $\underset{(A)}{\varphi}_m$ and $\underset{(A)}{\psi}_M$ are functions of x^0 and x^3. Then from (3.5) ~ (3.8) we obtain the following equations :

$$D(\underset{(A)}{\psi}_5+\underset{(A)}{\varphi}_2) = -2\Phi_0, \tag{6.14a}$$

$$D(\underset{(A)}{\psi}_2{}^*-\underset{(A)}{\varphi}_1{}^*) = [(g_2-g_3)/g_1]\Phi_0, \tag{6.14b}$$

$$\Delta(\underset{(A)}{\psi}_2{}^*-\underset{(A)}{\varphi}_1{}^*) = -2\Phi_2{}^*, \tag{6.15a}$$

$$\Delta(\underset{(A)}{\psi}_5+\underset{(A)}{\varphi}_2) = [(g_2-g_3)/g_1]\Phi_2{}^*, \tag{6.15b}$$

$$D(\underset{(A)}{\psi}_5-2\underset{(A)}{\varphi}_2) = \Delta\underset{(A)}{\psi}_0, \tag{6.16}$$

$$D\underset{(A)}{\psi}_7 = \Delta(\underset{(A)}{\psi}_2+2\underset{(A)}{\varphi}_1), \tag{6.17}$$

$$D\underset{(A)}{\psi}_6 = \Delta\underset{(A)}{\psi}_1, \tag{6.18a}$$

$$\underset{(A)}{\psi}_1 = 2\underset{(A)}{\varphi}_0, \quad \underset{(A)}{\psi}_6 = -2\underset{(A)}{\varphi}_3 \tag{6.18b,c}$$

$$\underset{(A)}{\psi}_3 = \underset{(A)}{\psi}_4 = 0. \tag{6.19}$$

Here notice that these equations are invariant under gauge transformations (3.9) and (3.10) which are now rewritten in dyad form as

$$\begin{aligned}
&\underset{(A)}{\tilde{\varphi}}_0 = \underset{(A)}{\varphi}_0-(1/3)D\lambda_1, && \underset{(A)}{\tilde{\varphi}}_1 = \underset{(A)}{\varphi}_1-(1/3)D\lambda_2,\\
&\underset{(A)}{\tilde{\varphi}}_2 = \underset{(A)}{\varphi}_2+(1/3)\Delta\lambda_0, && \underset{(A)}{\tilde{\varphi}}_3 = \underset{(A)}{\varphi}_3+(1/3)\Delta\lambda_1,\\
&\underset{(A)}{\tilde{\psi}}_0 = \underset{(A)}{\psi}_0-D\lambda_0, && \underset{(A)}{\tilde{\psi}}_1 = \underset{(A)}{\psi}_1-(2/3)D\lambda_1,\\
&\underset{(A)}{\tilde{\psi}}_2 = \underset{(A)}{\psi}_2-(1/3)D\lambda_2, && \underset{(A)}{\tilde{\psi}}_3 = \underset{(A)}{\psi}_3, \quad \underset{(A)}{\tilde{\psi}}_4 = \underset{(A)}{\psi}_4,\\
&\underset{(A)}{\tilde{\psi}}_5 = \underset{(A)}{\psi}_5-(1/3)\Delta\lambda_0, && \underset{(A)}{\tilde{\psi}}_6 = \underset{(A)}{\psi}_6-(2/3)\Delta\lambda_1,\\
&\underset{(A)}{\tilde{\psi}}_7 = \underset{(A)}{\psi}_7-\Delta\lambda_2,
\end{aligned}$$

where $\lambda_M(M = 0, 1, 2)$, which are dyad components of λ_{AB}, are now arbitrary functions of x^0 and x^3 only. Accordingly, we first find from (6.18a,b,c) that we can choose the parameter λ_1 in such a way that $\underset{(A)}{\psi}_1$ and $\underset{(A)}{\psi}_6$, and therefore $\underset{(A)}{\varphi}_0$ and $\underset{(A)}{\varphi}_3$ will vanish. And furthermore, we can see from (6.16) and (6.17) that the parameters λ_0 and λ_2 are chosen so as to realize the relations

$$\begin{aligned}
&\underset{(A)}{\psi}_0 = \underset{(A)}{\psi}_7 = 0,\\
&\underset{(A)}{\psi}_2+2\underset{(A)}{\varphi}_1 = 0, \quad \underset{(A)}{\psi}_5-2\underset{(A)}{\varphi}_2 = 0.
\end{aligned} \tag{6.20}$$

Then (6.14) and (6.15) become

$$D \underset{(A)}{\varphi_2} = -2/3\Phi_0,\ \Delta \underset{(A)}{\varphi_1}^* = 2/3\Phi_2^*, \tag{6.21a.b}$$
$$D \underset{(A)}{\varphi_1}^* = -[(g_2-g_3)/3g_1]\Phi_0, \tag{6.22a}$$
$$\Delta \underset{(A)}{\varphi_2} = [(g_2-g_3)/3g_1]\Phi_2^*. \tag{6.22b}$$

We require here that right- and left-moving plane waves (represented by Φ_2 and Φ_0, respectively) should be represented by the same potential function. Then we conclude from these equations that [A] when $g_1 = -g_2$, $g_4 = 0$ and $3g_1+g_3 = 0$, then we get

$$\underset{(A)}{\varphi_1}^* = \underset{(A)}{\varphi_2} \tag{6.23}$$

and that [B] when $g_1 = g_2$, $g_5 = 0$ and $3g_1-g_3 = 0$, then we get

$$\underset{(A)}{\varphi_1}^* = -\underset{(A)}{\varphi_2}. \tag{6.24}$$

In both cases [A] and [B] we obtain at once from (6.21a,b), as an integrability condition, the d'Alembert equation

$$\square \underset{(A)}{\varphi_1} = 0,$$

noting that the fields Φ_0 and Φ_2 are arbitrary functions of only x^0+x^3 and x^0-x^3, respectively. We thus see that the null field expressed by Φ_2 with the energy density (6.12) is represented by only one independent dyad component $\underset{(A)}{\varphi_1}$ of the irreducible spinors $\underset{(A)}{\varphi}_{\dot{A}B}$ and $\underset{(A)}{\psi}_{\dot{E}ABC}$ of Lorentz gauge field $A_{km\mu}$. Furthermore, noting that $\underset{(A)}{\varphi}_{\dot{A}B}$ is generally equivalent to two real vectors, we can see also that null field is represented for the cases [A] and [B] , respectively, by

$$\begin{cases} V^\mu = -3\cdot 2^{1/2}(0,\ \underset{(A)}{\varphi_1}+\underset{(A)}{\varphi_1}^*,\ i(\underset{(A)}{\varphi_1}-\underset{(A)}{\varphi_1}^*),\ 0), \\ A^\mu = (0,\ 0,\ 0,\ 0) \end{cases} \tag{6.25}$$

and

$$\begin{cases} V^\mu = (0,\ 0,\ 0,\ 0), \\ A^\mu = 2^{1/2}(0,\ i(\underset{(A)}{\varphi_1}-\underset{(A)}{\varphi_1}^*),\ -(\underset{(A)}{\varphi_1}+\underset{(A)}{\varphi_1}^*),\ 0). \end{cases} \tag{6.26}$$

Here V^μ and A^μ are respectively the irreducible vector and axial-vector components of the Lorentz gauge field $A_{km\mu}$. Their spinor equivalents $V_{\dot{A}B}$ and $A_{\dot{A}B}$ are defined in terms of $\underset{(A)}{\varphi}_{\dot{A}B}$ by

$$V_{\dot{A}B} = 3(\underset{(A)}{\varphi}_{\dot{A}B}+\underset{(A)}{\varphi}_{B\dot{A}}) \text{ and } A_{\dot{A}B} = -i(\underset{(A)}{\varphi}_{\dot{A}B}-\underset{(A)}{\varphi}_{B\dot{A}}),$$

respectively.

Thus, we find at once (or from (5.7)) that the field satisfying [A] has odd parity and the field satisfying [B] , even parity.

Finally, it should be remarked that the following condition is to added to make the energy density (6.12) positive definite in both cases [A] and [B]

$$g_1 > 0, \text{ i.e., } (2a_3 - a_5) < 0. \tag{6.27}$$

§ 7 . Concluding remark

In this paper we have investigated, using a spinor technique, the existence of a "null field" in PGT propagating with positive energy in vacuum under the condition $C_1 = C_2 = C_3 = 0$. As a result, we found the existence of such a field that has spin-parity 1^- or 1^+ in accordance with the parameter conditions [A] $g_1 = -g_2$, $3g_1 + g_3 = 0$ and $g_4 = 0$ or [B] $g_1 = g_2$, $3g_1 - g_3 = 0$ and $g_5 = 0$. Here it should be noticed that the additional restricting $g_1 > 0$ is required for both cases to make the energy density positive definite.

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Appendix : The matrix equivalent to a symmetric energy-momentum tensor T^{km}

$$T^{km} \to T_{\dot{A}B\dot{C}D} = \chi_{\dot{A}\dot{C}BD} + \lambda\varepsilon_{\dot{A}\dot{C}}\varepsilon_{BD}$$

$$\chi_{\dot{A}\dot{C}BD} = \chi_M \xi_{(M)\dot{A}\dot{C}BD} = \chi_{mn}\eta_{m\dot{A}B}\eta_{n\dot{C}D}$$

$$\chi_{mn} = (1/2)\begin{pmatrix} -(2^{1/2}x_2 + x_4) & x_1 + x_3 & -x_5 - x_7 & 2^{1/2}x_6 \\ x_1 + x_3 & 2^{1/2}x_0 + x_4 & 2^{1/2}x_8 & -x_5 + x_7 \\ -x_5 - x_7 & 2^{1/2}x_8 & -2^{1/2}x_0 + x_4 & x_1 - x_3 \\ 2^{1/2}x_6 & -x_5 + x_7 & x_1 - x_3 & 2^{1/2}x_2 - x_4 \end{pmatrix} \tag{A.1}$$

where

$$\begin{aligned} x_0 &= 2^{-1/2}(\chi_0 + \chi_8), & x_1 &= i(\chi_1 + \chi_7), \\ x_2 &= 2^{-1/2}(\chi_2 + \chi_6), & x_3 &= i(\chi_3 + \chi_5), \\ x_4 &= 2\chi_4, & x_5 &= \chi_3 - \chi_5, \\ x_6 &= i2^{-1/2}(\chi_2 - \chi_6), & x_7 &= \chi_1 - \chi_7, \\ x_8 &= i2^{-1/2}(\chi_0 - \chi_8). \end{aligned}$$

Incidentally, when $T_{(F)}{}^{(km)}$ is substituted for T^{km}, then we obtain for the energy density $\underset{(F)}{T^{00}}$

$$\underset{(F)}{T^{00}} = (1/2)(\underset{(F)}{\chi_0} + 2\underset{(F)}{\chi_4} + \underset{(F)}{\chi_8}) \tag{A.2}$$

because of $\lambda_{(F)} = 0$.

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