Notes on metrics and counting functions on B^n

Dedicated to Professor Kenji Nakagawa on his 70th birthday

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0. Introduction and preliminaries. Let $\boldsymbol{C}$ be the field of the complex numbers. Let $V = V^{1,n}(\boldsymbol{C})$ $(n \geqq 1)$ denote the vector space $\boldsymbol{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\overline{z_0^*}w_0^* + \sum_{k=1}^{n}\overline{z_k^*}w_k^*,$$

where $z^* = (z_0^*, z_1^*, \ldots, z_n^*)$ and $w^* = (w_0^*, w_1^*, \ldots, w_n^*)$ in V. An automorphism g of V will be called a unitary transformation. (g must be linear and $\Phi(g(z^*), g(w^*)) = \Phi(z^*, w^*)$ for all $z^*, w^* \in V$.) We denote the group of all unitary transformations by $U(1, n ; \boldsymbol{C})$.

Let $V_0 = \{z^* \in V|\ \Phi(z^*, z^*) = 0\}$ and $V_- = \{z^* \in V|\ \Phi(z^*, z^*) < 0\}$. Obviously V_0 and V_- are invariant under $U(1, n ; \boldsymbol{C})$. Let $\pi(V)$ be the projective space obtained from V. This is defined, as usual, by using the equivalence relation in $V-\{0\}$: $u^* \sim v^*$ if there exists $\lambda \in \boldsymbol{C}-\{0\}$ such that $u^* = \lambda v^*$. Let $\pi : V-\{0\} \rightarrow \pi(V)$ denote the projection map. We define : $H^n(\boldsymbol{C}) = \pi(V_-)$. Let $\overline{H^n(\boldsymbol{C})}$ denote the closure of $H^n(\boldsymbol{C})$ in the projective space $\pi(V)$. An element $g \in U(1, n ; \boldsymbol{C})$ operates in $\pi(V)$, leaving $\overline{H^n(\boldsymbol{C})}$ invariant. If $(z_0^*, z_1^*, \ldots, z_n^*) \in V_-$, the condition $-|z_0^*|^2 + \sum_{k=1}^{n}|z_k^*|^2 < 0$ implies that $z_0^* \neq 0$. Therefore we may define a set of coordinates $z = (z_1, z_2, \ldots, z_n)$ in $H^n(\boldsymbol{C})$ by $z_k(\pi(z^*)) = z_k^* z_0^{*-1}$. In this way $H^n(\boldsymbol{C})$ becomes identified with the complex unit ball $B^n = B^n(\boldsymbol{C}) = \{z = (z_1, z_2, \ldots, z_n) \in \boldsymbol{C}^n|\ \|z\|^2 = \sum_{k=1}^{n}|z_k|^2 < 1\}$. A unitary transformation is regarded as a transformation operating on $\overline{B^n}$. We use the same symbol $U(1, n ; \boldsymbol{C})$ to denote the group of all these transformations. Throughout this paper G will denote a discrete subgroup of $U(1, n ; \boldsymbol{C})$. We assume that the stabilizer G_0 of 0 consists only of the identity.

In this paper we shall show the properties of metrics on B^n in Section 1 and discuss a counting function $n(r, z)$ in Section 2.

1. Metrics d, $\boldsymbol{\delta}$ and $\boldsymbol{\delta_a}$. Let $d(\cdot)$ be the distance which is induced from the metric

$$g_{i\bar{j}}(z) = \delta_{ij}(1-\|z\|^2)^{-1} + \overline{z_i} z_j (1-\|z\|^2)^{-2},$$

where $z = (z_1, z_2, \ldots, z_n) \in B^n$. We recall that $d(z, w)$ is expressed as

$$d(z, w) = \cosh^{-1} [|\Phi(z^*, w^*)| \{\Phi(z^*, z^*)\Phi(w^*, w^*)\}^{-1/2}],$$

where $z^* \in \pi^{-1}(z)$ and $w^* \in \pi^{-1}(w)$. Set

$$\delta(z, w) = [1 - \{\Phi(z^*, z^*)\Phi(w^*, w^*)\} |\Phi(z^*, w^*)|^{-2}]^{1/2}$$

for $z, w \in B^n$ (see [3, p. 180]).

Proposition 1. 1. *Let* z and w be points in B^n.

(a) $\delta(z, w) = \tanh d(z, w)$.

(b) $\delta(g(z), g(w)) = \delta(z, w)$ *for any element* $g \in U(1, n ; \boldsymbol{C})$.

(c) $d(z, w) = (1/2) \log (1+\delta(z, w))(1-\delta(z, w))^{-1}$.

(d) $d(z, w) \geqq \delta(z, w)$.

Proof. (a) It is seen that

$$\begin{aligned}\tanh^2 d(z, w) &= 1 - \operatorname{sech}^2 d(z, w) \\ &= 1 - [\{\Phi(z^*, z^*)\Phi(w^*, w^*)\}^{-1/2} |\Phi(z^*, w^*)|]^{-2} \\ &= 1 - [\{\Phi(z^*, z^*)\Phi(w^*, w^*)\} |\Phi(z^*, w^*)|^{-2}] \\ &= \delta^2(z, w).\end{aligned}$$

Thus $\delta(z, w) = \tanh d(z, w)$.

(b) This follows from (a) and the invariance of d under $U(1, n ; \boldsymbol{C})$.

(c) This is immediate.

(d) By (a), $\delta(z, w) = \tanh d(z, w) \leqq d(z, w)$.■

Proposition 1. 2. *The function* δ *is a distance function on* B^n.

Proof. By (a) in Proposition 1. 1,

$$\begin{aligned}&\delta(z, w) \geqq 0 \text{ and } \delta(z, w) = 0 \rightleftarrows z = w ; \\ &\delta(z, w) = \delta(w, z).\end{aligned}$$

Therefore we have only to prove the triangle inequality. Let x, y and z be points in B^n. Using (a) in Proposition 1. 1 and the addition theorem on tanh, i. e.

$$\tanh\{d(x, y) + d(y, z)\} = \{\tanh d(x, y) + \tanh d(y, z)\} \{1 + \tanh d(x, y) \cdot \tanh d(y, z)\}^{-1},$$

we see that

$$\begin{aligned}\delta(x, y) + \delta(y, z) &= \tanh d(x, y) + \tanh d(y, z) \\ &= \tanh\{d(x, y) + d(y, z)\}\{1 + \tanh d(x, y) \cdot \tanh d(y, z)\} \\ &\geqq \tanh d(x, z) = \delta(x, z).\end{aligned}$$ ■

Set

$$\delta_\alpha(z, w) = [1-\{\Phi(z^*, z^*)\Phi(w^*, w^*)|\Phi(z^*, w^*)|^{-2}\}^\alpha]^{1/2}$$

and let

$$d_\alpha = (1/2)\log(1+\delta_\alpha)(1-\delta_\alpha)^{-1}.$$

It is easy to see that $\delta_1 = \delta$ and $d_1 = d$.

Proposition 1. 3.

(a) *The functions* δ_α *and* d_α *are increasing functions of* $\alpha > 0$.

(b) $\delta_\alpha = \tanh d_\alpha$.

(c) $\operatorname{sech} d_\alpha = \operatorname{sech}^\alpha d$.

(d) *If* $\alpha \in (0, 1)$, *then* d_α *is a distance function on* B^n.

Proof. (a) This is immediate.

(b) We see that

$$\tanh d_\alpha = \tanh\{(1/2)\log(1+\delta_\alpha)(1-\delta_\alpha)^{-1}\} = \delta_\alpha.$$

(c) The equality (a) in Proposition 1. 1 yields

$$\begin{aligned}\operatorname{sech}^2 d_\alpha &= 1-\tanh^2 d_\alpha = (1-\delta^2)^\alpha = (1-\tanh^2 d)^\alpha \\ &= \operatorname{sech}^{2\alpha} d.\end{aligned}$$

(d) We have only to prove the triangle inequality. Since $U(1, n ; \boldsymbol{C})$ is transitive on B^n, it is sufficient to show that

$$d_\alpha(z, w) \leqq d_\alpha(z, 0)+d_\alpha(w, 0) \qquad \text{for } z, w \in B^n. \tag{1}$$

Set $t_1 = \delta_\alpha(z, 0)$, $t_2 = \delta_\alpha(w, 0)$ and $t_3 = \delta_\alpha(z, w)$. Then (1) is equivalent to the following inequality :

$$(1+t_3)(1-t_3)^{-1} \leqq (1+t_1)(1-t_1)^{-1}(1+t_2)(1-t_2)^{-1}. \tag{2}$$

By (2), we obtain

$$t_3^2 \leqq \{(t_1+t_2)(1+t_1t_2)^{-1}\}^2.$$

From this it follows that

$$\begin{aligned}(1-t_3^2) &\geqq 1-\{(t_1+t_2)(1+t_1t_2)^{-1}\}^2 \\ &= (1-t_1^2)(1-t_2^2)(1+t_1t_2)^{-2}.\end{aligned} \tag{3}$$

We note that

$$1-\delta_\alpha(z, w)^2 = (1-\delta(z, w)^2)^\alpha. \tag{4}$$

By using (3) and (4), we have

$$1-\delta(z, w)^2 \geqq (1-\delta(z, 0)^2)(1-\delta(w, 0)^2)(1+t_1t_2)^{-2/\alpha}.$$

This implies that

$$\Phi(z^*, z^*)\Phi(w^*, w^*)|\Phi(z^*, w^*)|^{-2} \geqq \Phi(z^*, z^*)\Phi(w^*, w^*)(1+t_1t_2)^{-2/\alpha}$$

for $z^* = (1, z_1, z_2, \ldots, z_n) \in \pi^{-1}(z)$ and $w^* = (1, w_1, w_2, \ldots, w_n) \in \pi^{-1}(w)$. Therefore we have only to prove that

$$|\Phi(z^*, w^*)| \leqq (1+t_1t_2)^{1/\alpha}. \tag{5}$$

We can show that if $\alpha \in (0, 1)$, then the inequality (5) is true. In fact,

$$\begin{aligned}(1+t_1t_2)^{1/\alpha} &\geqq 1+(1/\alpha)t_1t_2 \\ &\geqq 1+(1/\alpha)[\{1-(1-\|z\|^2)^\alpha\}\{1-(1-\|w\|^2)^\alpha\}]^{1/2} \\ &\geqq 1+(1/\alpha)\{1-(1-\alpha\|z\|^2)\}^{1/2}\{1-(1-\alpha\|w\|^2)\}^{1/2} \\ &= 1+\|z\|\|w\| \geqq |\Phi(z^*, w^*)|.\end{aligned}$$

Thus our proof is complete.

Proposition 1. 4.

$$\delta(z, w) \leqq (\|z\|+\|w\|)(1+\|z\|\|w\|)^{-1} \qquad \textit{for } z, w \in B^n.$$

To prove Proposition 1. 4, we need a lemma.

Lemma 1. 5. *If* $0 \leqq r < 1$, *then a function* $f(x) = (r+x)(1+rx)^{-1}$ *is increasing in* $x \geqq 0$.

Proof of Proposition 1. 4. By (b) in Proposition 1. 1, we may assume that $z = (r, 0, \ldots, 0)$ and $w = (w_1, w_2, \ldots, w_n)$, where $0 \leqq r < 1$. Let $z^* = (1, r, 0, \ldots, 0) \in \pi^{-1}(z)$ and $w^* = (1, w_1, w_2, \ldots, w_n) \in \pi^{-1}(w)$. It follows from Lemma 1. 5 that

$$\begin{aligned}\delta(z, w)^2 &= 1-(1-r^2)(1-\|w\|^2)|1-rw_1|^{-2} \\ &\leqq 1-(1-r^2)(1-|w_1|^2)|1-rw_1|^{-2} \\ &= |r-w_1|^2|1-rw_1|^{-2} \\ &\leqq (r+|w_1|)^2(1+r|w_1|)^{-2} \\ &\leqq (r+\|w\|)^2(1+r\|w\|)^{-2}. \blacksquare\end{aligned}$$

Proposition 1.6. *Let* g *be an element of* $U(1, n\,;\, \boldsymbol{C})$. *For* $z, w \in B^n$

$$\|g(z)\| \leqq (\|z\|+\|g(0)\|)(1+\|z\|\|g(0)\|)^{-1}.$$

Proof. From (a) in Proposition 1. 1 it follows that

$$\|g(0)\| = \delta(0, g(0)) = \delta(g^{-1}(0), 0) = \|g^{-1}(0)\|.$$

By using this equality and Proposition 1. 4, we have

$$\begin{aligned}\|g(z)\| &= \delta(0, g(z)) \\ &= \delta(g^{-1}(0), z) \\ &\leqq (\|z\|+\|g^{-1}(0)\|)(1+\|z\|\|g^{-1}(0)\|)^{-1} \\ &\leqq (\|z\|+\|g(0)\|)(1+\|z\|\|g(0)\|)^{-1}. \blacksquare\end{aligned}$$

3. Counting function $n(r, z)$. Let $n(r, z) = \#\{g \in G | \ \|g(z)\| < r\}$.

Theorem 2. 1. *If* $\|z\| < r < 1$, *then*

$$n\left(\frac{r-\|z\|}{1-r\|z\|}, 0\right) \leqq n(r, z) \leqq n\left(\frac{r+\|z\|}{1+r\|z\|}, 0\right).$$

Proof. Proposition 1. 6 implies that

$$\{g \in G | \ \|g(z)\| < r\} \supset \{g \in G | \ \|g(0)\| < (r-\|z\|)(1-r\|z\|)^{-1}\}$$

for $\|z\| < r < 1$. If $\|g(0)\| < r$, then

$$\|g(z)\| \leqq (\|z\|+\|g(0)\|)(1+\|z\|\|g(0)\|)^{-1} \leqq (r+\|z\|)(1+r\|z\|)^{-1}$$

Therefore we have our desired inequalities.■

Let D_0 be the Dirichlet polyhedron for G centered at 0. We note that D_0 is expressed as

$$D_0 = \{z \in B^n | \ \|g_k(z)\| > \|z\| \ for \ all \ g_k \ in \ G-\{identity\}\}.$$

Let dV be the volume element which is induced from the metric $g_{i\bar{j}}$. It is easy to see that $dV(z) = (i/2)^n(1-\|z\|^2)^{-(n+1)}dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge \ d\bar{z}_n$ for $z = (z_1, \ldots, z_n) \in B^n$. We use $vol(A)$ for the volume of A measured by dV.

Theorem 2. 2. *Suppose that the volume of D_0 is finite. Let $a \in D_0$ and $\|a\| < \rho < 1$. There exists r_0 such that the following inequality is satisfied for $r_0 \leqq r \leqq 1$.*

$$A(1-r)^{-n} \leqq n(r,a) \leqq B(1-r)^{-n},$$

where A is a constant, which depends only on ρ and B is a numerical constant.

Proof. Let $g_0, g_1, \ldots$ be the complete list of elements in G. In virtue of [3, Proposition 4. 1] , we have only to prove that

$$A(1-r)^{-n} \leqq n(r,a).$$

We choose r_0 such that $\rho < r_0 < 1$. Let F_0 be the part of D_0 which lies outside of $\|z\| = r_0$, where we take $1-r_0$ so small that the volume of F_0 is less than ε. Let F_k be the image of F_0 under $g_k \in G$ and put $F = \bigcup_{k \geqq 0} F_k$. Denote $F \cap \{z | \ \|z\| < r\}$ by $F(r)$, where $r_0 < r < 1$. Then

$$\begin{aligned} vol(F(r)) &= \int_{F_0} n(r, z)dV(z) \\ &\leqq constant \cdot (1-r)^{-n} vol(F_0) \\ &\leqq constant \cdot \varepsilon(1-r)^{-n}. \end{aligned} \tag{6}$$

Put $H_0 = D_0 - F_0$ and let $\{H_k\}$ be its image under G and $H = \bigcup_{k \geqq 0} H_k$. Let $H(r) = H \cap \{z | \ \|z\| < r < 1\}$. It is seen that $F(r) \cup H(r) = \{z | \ \|z\| < r\}$ and $F(r) \cap H(r) = \phi$.

Therefore

$$vol(F(r))+vol(H(r)) = vol(\{z|\ \|z\| < r\}) \geqq constant\cdot(1-r)^{-n}. \tag{7}$$

It follows from (6) and (7) that

$$vol(H(r)) \geqq constant\cdot(1-r)^{-n}-constant\cdot\varepsilon(1-r)^{-n}. \tag{8}$$

Let $r_0{}^*$ be the δ-diameter of H_0 and set $r_1 = (r+r_0{}^*)(1+rr_0{}^*)^{-1}$. Using [3, Proposition 2. 1], we see that, if $H_k \cap \{z|\ \|z\| < r\} \neq \phi$, then H_k is included in $\{z|\ \|z\| < r_1\}$. Hence $H(r)$ is contained in $\bigcup_{k\geqq 0} H_k$, where

$$H_k \subset \{z|\ \|z\| < r_1\}. \tag{9}$$

Since $a \in D_0 \cap \{z|\ \|z\| < \rho\}$, the number of $\{H_k\}$ satisfying (9) is less than $n(a, r_1)$. Therefore we have

$$vol(H(r)) \leqq n(r_1, a)vol(H_0). \tag{10}$$

By (8) and (10),

$$n(r_1, a) \geqq constant\cdot(1-r)^{-n}(vol(H_0))^{-1}.$$

Noting that $(1-r_1)(1+r^*r_0{}^*)^{-1} = 1-r$, we obtain

$$n(r_1, a) \geqq A(1-r_1)^{-n}.$$

Writing r instead of r_1, we have

$$n(r, a) \geqq A(1-r)^{-n} \qquad \text{for } r_0 \leqq r < 1.$$

Thus our proof is complete.

References

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