ON GENERALIZED QUATERNIONIC UNITARY SPINORS

Hirohisa TACHIBANA

Graduate School of Science, Okayama University of Science,

Ridai-cho 1-1, Okayama 700, Japan

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ABSTRACT

We investigate the algebra of *generalized quaternionic unitary spinors* (GQUSs) which are two-component generalized quaternionic objects subjected to $U(2\,;\mathbf{H}_{(\lambda)})$ transformation, where $\mathbf{H}_{(\lambda)}$ is a set of *generalized quaternions* whose imaginary units $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ have the relations $\boldsymbol{i}^2 = -1$, $\boldsymbol{j}^2 = -\lambda$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$ and $\lambda = \pm 1$. We investigate the case of $\lambda = -1$ for which the isomorphism $U(2\,;\mathbf{H}_{(-1)}) \cong SO(3, 2)$ or anti-de Sitter group holds. Furthermore, in this case, we discuss a simple connection between GQUSs and a time-like geodesic in anti-de Sitter space-time.

1. INTRODUCTION

Since Dirac introduced the spinors to the relativistic quantum mechanical equations for electrons[1], many physicists studied the spinor analysis and their applications to physics[2]. Kugo and Townsend, to apply to the supersymmetry theory, extended the real and complex spinor algebra to the quaternionic spinor algebra[3].

Kugo and Townsend found the isomorphism $SO(5, 1) \cong SL(2, \mathbf{H})$ as well as well known isomorphisms $SO(2, 1) \cong SL(2, \mathbf{R})$ and $SO(3, 1) \cong SL(2, \mathbf{C})$, where we have the real numbers $\mathbf{R}$, the complex numbers $\mathbf{C}$ and the ordinary quaternions $\mathbf{H}$ which have the positive difinite norms. Furthermore, they found the isomorphisms $SO(5) \cong U(2\,;\mathbf{H})$ and $SO(4, 1) \cong U(1, 1\,;\mathbf{H})$ as well as the isomorphisms $SO(3) \cong SU(2)$ and $SO(2, 1) \cong SU(1, 1)$[4], respectively. Thus, the vectors in representation spaces of SL(2 ; **H**), U(2 ; **H**) and U(1, 1 ; **H**) are the spinors in the 5 + 1-dimensional Minkowski space-time, the 5-dimensional Euclidian space and the 4 + 1-dimensional Minkowski space-time, respectively. Particularly, since the group SO(4, 1) is sometimes called *de Sitter (dS) group,* we shall call the U(1, 1 ; **H**)-spinors *dS-spinors.*

Resently, in the theory of confiniment of quarks and gluons, *anti-de Sitter (AdS)* space-time rather than dS space-time has been called attention, where the quarks and gluons move inside a spherical bag with AdS metric[5]. Furthermore, Dullemond, Rijken and van Beveren suggested the connection between QCD and SO(3, 2) which is sometimes called *AdS group* via spontaneous symmetry breaking of SO(4, 2) symmetry of QCD Lagrangian to SO(3, 2) symmetry[6].

From the above facts, we are interested in the study of the spinors in the 3 + 2-dimensional Minkowski space-time or *AdS-spinors* rather than dS-spinors. However, by Kugo and Townsend's extension of the spinors, AdS-spinors cannot be derived.

In this paper, we investigate the AdS-spinors. Here, instead of ordinary quaternions **H**, we use *generalized quaternions* $\mathbf{H}_{(\lambda)}$[7] which have the signature $(+\ +\ \pm\ \pm)$ of the square norms defined by λ of $\mathbf{H}_{(\lambda)}$ where

$$\lambda = \pm 1 \tag{1-1}$$

and we shall describe U(2 ; $\mathbf{H}_{(\lambda)}$) -spinors or *generalized quaternionic unitary spinors (CQUSs).* If $\lambda = +1$, then GQUSs become the spinors in 5-dimensional Euclidian space, which are made clear by Kugo and Townsend. However, if $\lambda = -1$, GQUSs become the AdS-spinors. By GQUSs, we may not only express the spinors in 5-dimensional Euclidian space but also in 3 + 2-dimensional Minkowski space-time.

2. ALGEBRA OF 2-DIMENSIONAL GENERALIZED QUATERNIONIC MATRICES

We define an element of generalized quaternions $\mathbf{H}_{(\lambda)}$ by

$$\boldsymbol{q} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3, \qquad q_0,\ q_1,\ q_2,\ q_3 \in \mathbf{R} \tag{2-1a}$$

and its quaternion conjugate by

$$\bar{\boldsymbol{q}} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3, \tag{2-1b}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the quaternion imaginary units which have the relations ;

$$i^2 = -1, \qquad j^2 = -\lambda, \qquad ij = -ji = k, \tag{2-2}$$

and λ is defined by (1-1). We must note that generally $pq \neq qp$ and $\overline{pq} = \bar{q}\bar{p}$ for $\in \mathbf{H}_{(\lambda)}$. In addition, we denote the real part of $q \in \mathbf{H}_{(\lambda)}$ by $\mathrm{Re}(q)$, that is,

$$\mathrm{Re}(q) = \mathrm{Re}(\bar{q}) = \frac{1}{2}(q + \bar{q}), \qquad q \in \mathbf{H}_{(\lambda)} \tag{2-3}$$

Note that $\mathrm{Re}(qp) = \mathrm{Re}(pq)$ for $p,\ q \in \mathrm{H}_{(\lambda)}$.

Furthermore, we denote the square norm of $q \in \mathbf{H}_{(\lambda)}$ by $N(q)$, that is, for q of (2-1a) or (2-1b)

$$N(q) = N(\bar{q}) = \bar{q}\, q = q\, \bar{q} = q_0^2 + q_1^2 + \lambda q_2^2 + \lambda q_3^2 \in \mathbf{R}. \tag{2-4}$$

The square norms N of $\mathbf{H}_{(\lambda)}$ have an impotant property as follows;

$$N(pq) = N(p)N(q), \qquad p,\ q \in \mathbf{H}_{(\lambda)}. \tag{2-5}$$

A 2-dimensional generalized quaternionic vector (2-GQV) is a two-component object;

$$\phi = (\phi_A) = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad p,\ q \in \mathbf{H}_{(\lambda)}, \tag{2-6}$$

where A = 1, 2. For two 2-GQVs $\phi,\ \chi$ and $r \in \mathbf{H}_{(\lambda)}$, we define the following rules;

$$\phi + \chi = (\phi_A + \chi_A), \qquad r\phi = (r\phi_A) \neq \phi r = (\phi_A r). \tag{2-7}$$

A 2-dimensional generalized quaternionic matrix (2-GQM) is a four-component quaternionic object;

$$\mathbf{M} = (\mathbf{M}_{AB}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a,\ b,\ c,\ d \in \mathbf{H}_{(\lambda)}. \tag{2-8}$$

For two 2-GQMs $\mathbf{M},\ \mathbf{N}$ and $r \in \mathbf{H}_{(\lambda)}$, we define the following rules;

$$\mathbf{M} + \mathbf{N} = (\mathbf{M}_{AB} + \mathbf{N}_{AB}), \qquad r\mathbf{M} = (r\mathbf{M}_{AB}) \neq \mathbf{M}r = (\mathbf{M}_{AB}r). \tag{2-9}$$

Furthermore, for two 2-GQVs $\phi,\ \chi$ and two 2-GQMs $\mathbf{M},\ \mathbf{N}$, the products are defined as follows;

$$\phi^+\chi = \bar{\phi}_A \chi_A \in \mathbf{H}_{(\lambda)}, \tag{2-10a}$$

$$\mathbf{M}\phi = (\mathbf{M}_{AB}\phi_B)\ ;\ \text{a 2-GQV}, \tag{2-10b}$$

$$\phi\chi^+ = (\phi_A \bar{\chi}_B)\ ;\ \text{a 2-GQM}, \tag{2-10c}$$

$$\mathbf{MN} = (\mathbf{M}_{AC}\mathbf{N}_{CB})\ ;\ \text{a 2-GQM}, \tag{2-10d}$$

where the symbol $^+$ indicates the quaternion hermitian conjugate;

$$\phi^+ = (\bar{p} \quad \bar{q}) \qquad \text{for } \phi = \begin{pmatrix} p \\ q \end{pmatrix}, \qquad p, q \in \mathbf{H}_{(\lambda)}, \tag{2-11a}$$

$$\mathbf{M}^+ = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \text{ for } \mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbf{H}_{(\lambda)}. \tag{2-11b}$$

In addition, for two 2-GQVs ϕ, χ and two 2-GQMs $\mathbf{M}$, $\mathbf{N}$, we may easily prove the following equations;

$$(\mathbf{M}\phi)^+ = \phi^+\mathbf{M}^+, \qquad (\phi\chi^+)^+ = \chi\phi^+ \qquad (\mathbf{MN})^+ = \mathbf{N}^+\mathbf{M}^+. \tag{2-12}$$

Lemma 1: For a 2-GQM $\mathbf{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $a, b, c, d \in \mathbf{H}_{(\lambda)}$, if there exsist an inverse 2-GQM $\mathbf{M}^{-1}$ which statisfies the relation

$$\mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2-13}$$

then the $\mathbf{M}^{-1}$ can be written as follows;

$$\mathbf{M}^{-1} = \frac{1}{\det\mathbf{M}} \begin{pmatrix} N(d)\bar{a} - \bar{c}b\bar{d} & N(b)\bar{c} - \bar{a}b\bar{d} \\ N(c)\bar{b} - \bar{d}c\bar{a} & N(a)\bar{d} - \bar{b}a\bar{c} \end{pmatrix}, \tag{2-14}$$

$$\det\mathbf{M} = N(ad) + N(bc) - 2\mathrm{Re}\,(\bar{a}\, b\, \bar{d}\, c) \in \mathbf{R}. \tag{2-15}$$

Then, this 2-GQM $\mathbf{M}$ is called regular.

The proof of this lemma can be given by a direct compution of $\mathbf{M}^{-1}\mathbf{M}$ or $\mathbf{M}\mathbf{M}^{-1}$. In addition, we may easily prove the following expression;

$$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1} \tag{2-16}$$

for two regular 2-GQMs $\mathbf{M}$ and $\mathbf{N}$.

3. GENERALIZED QUATERNIONIC UNITARY SPINORS

A generalized quaternionic unitary spinor (GQUS) is a 2-GQV which transforms as follows;

$$\xi \longrightarrow \mathbf{U}\xi, \tag{3-1}$$

where $\mathbf{U}$ is a 2-dimensional generalized·quaternionic unitary matrix (2-GQUM) which is a regular 2-GQM statisfing the condition

$$\mathbf{U}^+\mathbf{U} = \mathbf{U}\mathbf{U}^+ = \mathbf{1} \text{ or } \mathbf{U}^+ = \mathbf{U}^{-1}. \tag{3-2}$$

Using lemma 1 and (3-2), any 2-GQUMs can be written in one of the following two forms;

$$\begin{cases} \mathbf{U} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ -\boldsymbol{d}\,\bar{\boldsymbol{b}}\,\boldsymbol{a}/N(\boldsymbol{a}) & \boldsymbol{d} \end{pmatrix}, \quad \boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{d} \in \mathbf{H}_{(\lambda)}, \\ N(\boldsymbol{a}) = N(\boldsymbol{d}) \neq 0, \qquad N(\boldsymbol{a}) + N(\boldsymbol{b}) = 1, \end{cases} \tag{3-3 a}$$

or

$$\begin{cases} \mathbf{U} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & -\boldsymbol{c}\,\bar{\boldsymbol{a}}\,\boldsymbol{b} \end{pmatrix}, \quad \boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c} \in \mathbf{H}_{(\lambda)}, \\ N(\boldsymbol{a}) = 0, \quad N(\boldsymbol{b}) = N(\boldsymbol{c}) = 1. \end{cases} \tag{3-3 b}$$

Since, from (3-3a) or (3-3b), a 2-GQUM has three quarternions satisfying but also two conditions, a 2-GQUM has ten independent real parameters. Thus, $U(2\,;\mathbf{H}_{(\lambda)})$ consisting of all 2-GQUMs forms a ten-parameter group.

The quantity

$$\xi^{+}\eta = \bar{\xi}_{A}\eta_{A} \text{ for two GQUSs } \xi \text{ and } \eta \tag{3-4}$$

is an invariant generalized quaternion under the $U(2\,;\mathbf{H}_{(\lambda)})$ transformation. Note here that, if we put $\eta = \xi$ in (3-4), then $\xi^{+}\xi$ is a real number. Particularly, if

$$\xi^{+}\xi = 0 \quad \text{for a non-zero GQUS } \xi, \tag{3-5}$$

then ξ is called *null* if and only if $\lambda = -1$.

A GQM Ξ is called GQUS of valence [1, 1] ([1, 1]-GQUS), if it transforms under the $U(2\,;\mathbf{H}_{(\lambda)})$ transformation similar to a product of two GQUSs ξ and η ; $\xi\eta^{+}$, that is,

$$\Xi \longrightarrow \mathbf{U}\Xi\mathbf{U}^{-1}, \quad \mathbf{U} \in U(2\,;\mathbf{H}_{(\lambda)}). \tag{3-6}$$

Lemma 2: If a [1, 1]-GQUS Θ is a hermitian ;

$$\Theta^{+} = \Theta, \tag{3-7}$$

then $\mathbf{U}\Theta\mathbf{U}^{-1}$ for $\mathbf{U} \in U(2\,;\mathbf{H}_{(\lambda)})$ is also a hermitian.

Proof: From (2-12c), (3-2) and (3-7), $(\mathbf{U}\Theta\mathbf{U}^{-1})^{+} = (\mathbf{U}\Theta\mathbf{U}^{+})^{+} = \mathbf{U}\Theta\mathbf{U}^{+} = \mathbf{U}\Theta\mathbf{U}^{-1}$.

The quantity

$$\mathrm{Re}(\mathrm{tr}\Xi) = \mathrm{Re}(\Xi_{AA}) \text{ for a [1. 1]-GQUSs } \Xi \tag{3-8}$$

is an invariant real number under the $U(2\,;\mathbf{H}_{(\lambda)})$ transformation, since $\mathrm{Re}(\mathrm{tr}\Xi) \longrightarrow \mathrm{Re}[\mathrm{tr}(\mathbf{U}\Xi\mathbf{U}^{-1})] = \mathrm{Re}(\mathbf{U}_{AB}\Xi_{BC}\mathbf{U}^{-1}{}_{CA}) = \mathrm{Re}(\mathbf{U}^{-1}{}_{CA}\mathbf{U}_{AB}\Xi_{BC}) = \mathrm{Re}[\mathrm{tr}(\mathbf{U}^{-1}$

$\mathbf{U}\Xi)] = \mathrm{Re}(\mathrm{tr}\Xi)$ for $\mathbf{U} \in \mathrm{U}(2;\mathbf{H}_{(\lambda)})$.

Lemma 3: If a [1, 1]-GQUS Θ is hermitian; $\Theta = \Theta^+$, then

$$\mathrm{tr}\Theta = \Theta_{\mathrm{AA}} \tag{3-9}$$

is an invariant real number under the $\mathrm{U}(2;\mathbf{H}_{(\lambda)})$ transformation.

The proof of this lemma is obvious since the diagonal components of Θ; Θ_{11} and Θ_{22}, are both real numbers. However, we must note that $\mathrm{tr}\Xi$ for a general [1, 1]-GQUS Ξ is not an invariant quantity under the $\mathrm{U}(2;\mathbf{H}_{(\lambda)})$ transformation, because of noncommutativity of Ξ_{11} and Ξ_{22}.

We shall close this section with a follwing definition. We denote the trace free part of a hermitian [1, 1]-GQUS Θ by $:\Theta:$, that is,

$$:\Theta: = \Theta - \frac{1}{2}(\mathrm{tr}\Theta)\mathbf{1}. \tag{3-10}$$

4. VECTORS IN 5-DIMENSIONAL EUCLIDIAN SPACE AND 3 + 2-DIMENSIONAL MINKOWSKI SPACE-TIME

Let us consider a hermitian trace free [1, 1]-GQUS; $\mathbf{V} = :\mathbf{V}:$. Then, $\mathbf{V}$ can be written as follows;

$$\mathbf{V} = v^{\mathrm{a}}\mathbf{E}_{\mathrm{a}}, \quad v^{\mathrm{a}} \in \mathbf{R}, \tag{4-1}$$

where a = 1, 2, 3, 4, 5 and

$$\mathbf{E}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{E}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\boldsymbol{i} \\ \boldsymbol{i} & 0 \end{pmatrix}, \quad \mathbf{E}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathbf{E}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\boldsymbol{j} \\ \boldsymbol{j} & 0 \end{pmatrix}, \quad \mathbf{E}_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\boldsymbol{k} \\ \boldsymbol{k} & 0 \end{pmatrix} \tag{4-2}$$

are five hermitian trace free [1, 1]-GQUSs which satisfy the following relations;

$$\mathbf{E}_{\mathrm{a}}\mathbf{E}_{\mathrm{b}} + \mathbf{E}_{\mathrm{b}}\mathbf{E}_{\mathrm{a}} = g_{\mathrm{ab}}\mathbf{1}, \tag{4-3}$$

$$g_{\mathrm{ab}} = \mathrm{diag}(1, 1, 1, \lambda, \lambda,) \tag{4-4}$$

From lemma 2 and lemma 3, $\mathbf{E}_{\mathrm{a}}$ transforms under the $\mathrm{U}(2;\mathbf{H}_{(\lambda)})$ transformation as follows;

$$\mathbf{E}_{\mathrm{a}} \longrightarrow \mathbf{U}\mathbf{E}_{\mathrm{a}}\mathbf{U}^{-1} = \mathbf{E}_{\mathrm{b}}L^{\mathrm{b}}{}_{\mathrm{a}}, \ \mathbf{U} \in \mathrm{U}(2;\mathbf{H}_{(\lambda)}), \ L^{\mathrm{a}}{}_{\mathrm{b}} \in \mathbf{R}. \tag{4-5}$$

Substituting $\mathbf{U}\mathbf{E}_{\mathrm{a}}\mathbf{U}^{-1} = \mathbf{E}_{\mathrm{b}}L^{\mathrm{b}}{}_{\mathrm{a}}$ for $\mathbf{E}_{\mathrm{a}}$ in (4-3), we may obtain for $L^{\mathrm{a}}{}_{\mathrm{b}}$ the following relations;

$$g_{\mathrm{cd}}L^{\mathrm{c}}{}_{\mathrm{a}}L^{\mathrm{d}}{}_{\mathrm{b}} = g_{\mathrm{ab}}. \tag{4-6}$$

The expression (4-6) is the condition for an element $L^{\mathrm{a}}{}_{\mathrm{b}}$ of $\mathrm{SO}(4+\lambda, 1-\lambda)$.

Thus, we see that the $U(2;\mathbf{H}_{(\lambda)})$ transformation induces the $SO(4+\lambda, 1-\lambda)$ transformation;

$$U(2;\mathbf{H}_{(\lambda)}): v^a \longrightarrow L^a{}_b v^b, \quad L^a{}_b \in SO(4+\lambda, 1-\lambda) \qquad (4\text{-}7)$$

for $\mathbf{V}$ of (4-1), since, from (4-5), we have $\mathbf{V} = v^a\mathbf{E}_a \longrightarrow \mathbf{UVU}^{-1} = v^a\mathbf{UE}_a\mathbf{U}^{-1} = L^a{}_b v^b \mathbf{E}_a$. In the case of $\lambda = -1$, $SO(4+\lambda, 1-\lambda)$ or $SO(3, 2)$ is called anti-de Sitter group, and then we shall call GQUSs AdS-spinors.

In consequence, v^a in $\mathbf{V}$ of (4-1) are the components of a vector $\mathbf{V}$ in 5-dimensional Euclidian space for $\lambda = -1$ or $3+2$-dimensional Minkowski space-time for $\lambda = -1$. Namely, we may identify a trace free hermitian [1, 1]-GQUS with a vector in the 5-dimensional Euclidian space or the $3+2$-dimensional Minkowski space-time. Therefore, we shall call the trace free hermitian [1, 1]-GQUSs 5-vectors and $\mathbf{E}_a$ of (4-2) the basis vectors.

The inner product $\langle\ ,\ \rangle$ for two 5-vectors can be written as follows;

$$\langle\mathbf{A}, \mathbf{B}\rangle = \mathrm{Re}[\mathrm{tr}(\mathbf{AB})] \text{ for two 5-vectors } \mathbf{A} \text{ and } \mathbf{B}, \qquad (4\text{-}8)$$

since, from (4-3),

$$\langle\mathbf{E}_a, \mathbf{E}_b\rangle = \mathrm{Re}[\mathrm{tr}(\mathbf{E}_a\mathbf{E}_b)] = g_{ab} \qquad (4\text{-}9)$$

for two basis vectors $\mathbf{E}_a$ and $\mathbf{E}_b$. Of course, for (4-8), we may easily prove the following relations;

$$\langle\mathbf{A}, \mathbf{B}\rangle = \langle\mathbf{B}, \mathbf{A}\rangle,\ \langle\mu\mathbf{A}, \mathbf{B}\rangle = \mu\langle\mathbf{A}, \mathbf{B}\rangle,\ \langle\mathbf{A}+\mathbf{B}, \mathbf{C}\rangle = \langle\mathbf{A}, \mathbf{C}\rangle + \langle\mathbf{B}, \mathbf{C}\rangle \qquad (4\text{-}10)$$

for $\mu \in \mathbf{R}$ and three 5-vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

The quantity

$$\mathbf{X} = :\xi\,\xi^+: \text{ for a non-zero GQUS } \xi \qquad (4\text{-}11a)$$

is a 5-vector; if we write $\mathbf{X}$ as

$$\mathbf{X} = x^a\mathbf{E}_a, \quad x^a \in \mathbf{R}, \qquad (4\text{-}11b)$$

then x^a can be written as follows;

$$\begin{aligned}
x^1 &= \sqrt{2}(\xi_{10}\xi_{20} + \xi_{11}\xi_{21} + \lambda\xi_{12}\xi_{22} + \lambda\xi_{13}\xi_{23}),\\
x^2 &= \sqrt{2}[\xi_{10}\xi_{21} - \xi_{11}\xi_{20} + \lambda(\xi_{12}\xi_{23} - \xi_{13}\xi_{22})],\\
x^3 &= \frac{1}{\sqrt{2}}(N(\xi_1) - N(\xi_2)), \qquad (4\text{-}11c)\\
x^4 &= \sqrt{2}(\xi_{10}\xi_{22} - \xi_{12}\xi_{20} + \xi_{13}\xi_{21} - \xi_{11}\xi_{23}),\\
x^5 &= \sqrt{2}(\xi_{10}\xi_{23} - \xi_{13}\xi_{20} + \xi_{11}\xi_{22} - \xi_{12}\xi_{21}),
\end{aligned}$$

where

$$\xi_A = \xi_{A0} + i\xi_{A1} + j\xi_{A2} + k\xi_{A3}. \qquad (4\text{-}11d)$$

The square length of this 5-vector **X** is given as follows;

$$\langle \mathbf{X}, \mathbf{X} \rangle = g_{ab}x^a x^b = \frac{1}{2}(\xi^+\xi)^2 \geq 0 \qquad (4\text{-}12)$$

Here, note that we have $\langle \mathbf{X}, \mathbf{X} \rangle = 0$ if and only if ξ is a null AdS-spinor.

5. SIMPLE CONNECTION BETWEEN TIME-LIKE GEODESIC IN ANTI-de SITTER SPACE-TIME AND NULL ANTI-de SITTER SPINORS

The AdS space-time $\mathbf{H}_1{}^4$ is a hyperboloid in $3+2$-dimensional Minkowski space-time $\mathbf{R}_2{}^5$, that is,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 - (x^5)^2 = -1^{8)}, \qquad x^a \in \mathbf{R}. \qquad (5\text{-}1\ a)$$

If we denete these quintuples $\{x^a\}$ by a 5-vector

$$\mathbf{X} = x^a \mathbf{E}_a, \qquad (5\text{-}2)$$

then the hyperboloid condition (5-1a) can be written as

$$\langle \mathbf{X}, \mathbf{X} \rangle = -1. \qquad (5\text{-}1\ b)$$

Let us consider a curve $\mathbf{X}_t$ through a point P in $\mathbf{H}_1{}^4$, where $\mathbf{X}_t$ have a real parameter t. If **V** is a tangent vector of $\mathbf{X}_t$ at the point P, then **V** is geven by

$$\mathbf{V} = \mathrm{d}\mathbf{X}_t/\mathrm{d}t \mid_P. \qquad (5\text{-}3)$$

Note that, from (5-1b) and (5-3), **V** satisfies the condition

$$\langle \mathbf{V}, \mathbf{X}_P \rangle = 0, \qquad (5\text{-}4)$$

where $\mathbf{X}_P$ is a 5-vector indicating the point P. Then, the equation for the geodesic passing through the point P in $\mathbf{H}_1{}^4$ can be written as follows;

$$[(\mathrm{d}/\mathrm{d}t)^2 - \langle \mathbf{V}, \mathbf{V} \rangle]\mathbf{X}_t = 0. \qquad (5\text{-}5)$$

The time-like solution of (5-5) is

$$\mathbf{X}_t = \mathbf{X}_P \cos t + \mathbf{V} \sin t \ \text{ for } \langle \mathbf{V}, \mathbf{V} \rangle = -1. \qquad (5\text{-}6)$$

This solution is physically important, since the massive particles in $\mathbf{H}_1{}^4$ move along the geodesic (5-6).

Now, we shall investigate a relation between the time-like geodesic and AdS-spinors. Let us consider the following two null AdS-spinors.

$$\iota = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ j \end{pmatrix}, \quad \kappa = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ k \end{pmatrix}, \qquad (5\text{-}7)$$

Then, we may define the following two null 5-vectors;

$$\mathbf{I} = \iota\,\iota^+ = :\iota\,\iota^+:, \quad \mathbf{K} = \kappa\,\kappa^+ = :\kappa\,\kappa^+:. \qquad (5\text{-}8)$$

A linear combination of **I** and **K**;

$$\mathbf{X}_P=\frac{1}{\sqrt{2}}(\mathbf{I}-\mathbf{K})=\mathbf{E}_4 \tag{5-9}$$

is a 5-vector which indicates a point in $\mathbf{H}_1{}^4$, since $\langle \mathbf{E}_4, \mathbf{E}_4\rangle = -1$. If we take this $\mathbf{X}_\mathrm{P}$ as a primary point $P \in \mathbf{H}_1{}^4$, then the 5-vector

$$\mathbf{V} = \underset{(4)}{\mathbf{U}}\,\mathbf{E}_5\underset{(4)}{\mathbf{U}}^{-1},\ \underset{(4)}{\mathbf{U}} \in \mathrm{U}(2\,;\mathbf{H}_{(-1)}), \tag{5-10a}$$

where $\underset{(4)}{\mathbf{U}}$ satisfies the condition

$$\underset{(4)}{\mathbf{U}}\,\mathbf{E}_4 = \mathbf{E}_4\underset{(4)}{\mathbf{U}}, \tag{5-10b}$$

is a time-like tangent vector at the point P, since $\langle \mathbf{V}, \mathbf{V}\rangle = \langle \mathbf{E}_5, \mathbf{E}_5\rangle = -1$ and $\langle \mathbf{V}, \mathbf{X}_\mathrm{P}\rangle = \langle \mathbf{E}_5, \mathbf{E}_4\rangle = 0$. Substituting (5-9) and (5-10a) into (5-6), and using (5-10b) and (5-9), we obtain

$$\begin{aligned}\mathbf{X}_\mathrm{t} &= \mathbf{E}_4\cos t + \underset{(4)}{\mathbf{U}}\,\mathbf{E}_5\underset{(4)}{\mathbf{U}}^{-1}\sin t\\ &= \underset{(4)}{\mathbf{U}}\exp(\boldsymbol{i}t/2)\mathbf{X}_\mathrm{P}\exp(-\boldsymbol{i}t/2)\underset{(4)}{\mathbf{U}}^{-1}\\ &= \underset{(4)}{\mathbf{U}}\exp(\boldsymbol{i}t/2)\frac{1}{\sqrt{2}}(\mathbf{I}-\mathbf{K})\exp(-\boldsymbol{i}t/2)\underset{(4)}{\mathbf{U}}^{-1}\end{aligned} \tag{5-11}$$

Similarly to (5-8) and (5-9), if we put

$$\mathbf{I}_\mathrm{t} = \iota_\mathrm{t}\,\iota_\mathrm{t}^{+},\quad \mathbf{K}_\mathrm{t} = \varkappa_\mathrm{t}\,\varkappa_\mathrm{t}^{+} \tag{5-12}$$

and

$$\mathbf{X}_t=\frac{1}{\sqrt{2}}(\mathbf{I}_t-\mathbf{K}_t), \tag{5-13}$$

then, from (5-11), (5-7) and (2-2), we may obtain

$$\begin{aligned}\iota_\mathrm{t} &= \underset{(4)}{\mathbf{U}}[\iota\cos(t/2) + \varkappa\sin(t/2)]\\ \varkappa_\mathrm{t} &= \underset{(4)}{\mathbf{U}}[-\iota\sin(t/2) + \varkappa\cos(t/2)]\end{aligned} \tag{5-14}$$

Here, of caurse, ι_t and $\varkappa_t$ are both null AdS-spinors. Differentiating these solutions (5-14) with respect to the parameter t, we obtain the following simultaneous differential equations;

$$\begin{cases} d\iota_\mathrm{t}/dt = \varkappa_\mathrm{t}/2\\ d\varkappa_\mathrm{t}/dt = -\iota_\mathrm{t}/2.\end{cases} \tag{5-15}$$

Let us denote the primary values the primary values of ι_t and $\varkappa_t$ in (5-15) by ι_P and $\varkappa_\mathrm{P}$, respectively, that is,

$$\iota_\mathrm{P} = \iota_\mathrm{t}\,\big|_{\,\mathrm{t=0}},\quad \varkappa_\mathrm{P} = \varkappa_\mathrm{t}\,\big|_{\,\mathrm{t=0}}. \tag{5-16}$$

If we impose on these on these ι_P and $\varkappa_\mathrm{P}$ the conditions

and

$$\iota_P{}^+\iota_P = 0, \quad \kappa_P{}^+\kappa_P = 0 \tag{5-17a}$$

$$\mathrm{Re}(\iota_P{}^+\kappa_P) = 0 \tag{5-17b}$$

then we may obtain

$$\mathbf{X}_t = \mathbf{X}_P\cos t + \mathbf{V}\sin t, \tag{5-18}$$

where

$$\mathbf{X}_t = \frac{1}{\sqrt{2}}(\iota_t\iota_t{}^+ - \kappa_t\kappa_t{}^+) \tag{5-19a}$$

and

$$\mathbf{X}_P = \frac{1}{\sqrt{2}}(\iota_P\iota_P{}^+ - \kappa_P\kappa_P{}^+), \quad \mathbf{V} = \frac{1}{\sqrt{2}}(\iota_P\kappa_P{}^+ + \kappa_P\iota_P{}^+). \tag{5-19b}$$

Using the conditions (5-17a) and (5-17b), and from (5-19), $\langle \mathbf{X}_P, \mathbf{X}_P\rangle = -1$, $\langle \mathbf{V}, \mathbf{V}\rangle = -1$ and $\langle \mathbf{V}, \mathbf{X}_P\rangle = 0$, we find that $\mathbf{X}_P$ is a 5-vector indicating a point $P \in \mathbf{H}_1{}^4$ and $\mathbf{V}$ is a time-like tangent vector at the point P. Comparing (5-18) with (5-6), we find that the curve $\mathbf{X}_t$ of (5-18) is a time-like geodesic in $\mathbf{H}_1{}^4$. Thus, we may see that the simultaneous differential equations (5-15) relate the time-like geodesic in $\mathbf{H}_1{}^4$ to the null AdS-spinors.

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