Solution of an Octonion Wave Equation and the Theory of Functions of a Split (Complex) Octonion Variable

K. Imaeda,* Mari Imaeda,** Susumu Ohta*** and H. Tachibana*

*Department of Electronic Engineering, Okayama University of Science, Ridaicho, Okayama, 700, Japan.

**IFUP, 31700, Cheswick Place, Ohio, 44139, U.S.A.

*** 1-11, Onaricho, Okayamashi, 703, Japan.

(Received September 30, 1986)

Abstract

We discuss the solution of the octoion wave equation and Proca's equation using the regular functions of an octonion variable developed before. Also, we discuss the extention of Proca's equation having four dimensional internal mass.

1. Introduction

Recently, an attention has been directed to octonions in theoretical physics in the hope of extending the 3 + 1 space-time frame work of the theory to 8 dimension to accommodate ever increasing quantum numbers assigned to elementary particles and fields. As an extension of the application of quaternions to Maxwell's equations(1), and Dirac-Clifford algebras to Dirac equation, an octonion field equation(2) was discussed in relation to the extra degree of freedom of octonions to connect the internal symmetry of the elementary particles and the higher dimensions needed for the unification of physical forces in the gauge theories.

Günaydin and Gürsey(3) formulated the quark structure using split octonion algebra and others discussed various attempts of unifying all the physical forces by using higher exceptional groups which essentially related to octonions.(4)

While in the scheme of unification of all physical forces: the 11 dimensional

Kaluza-Klein theory, octonions played an important role in the derivation of solutions of Cartan-Schouten-Englart equation [5], the torsion tensor on the S^7 sphere [6]. While Dündarer et al. [7] used the theory of octonion functions to derive the winding numbers.

Penney [2] had proposed an octonion wave equation and discussed a special case of the mass operator by reducing the extra degree of freedom: mass space to a scalar constant mass. He showed that the octonion wave equation reduces to a pair of Dirac equations when the 4 extra degree of freedom of mass space is reduced to a scalar constant mass and the extra degree of freedom is postulated to be the internal symmetry of isospin: proton and neutron. Quite recently, Joshi [8] had discussed the same octonion wave equation as Penney's. He showed that an octonion wave function has four dimensional internal space of mass and the internal symmetry is interpreted as the generation of the four families of fermions within the quark-lepton symmetry.

On the other hand, the theory of functions of an octonion variable is developed independently of field equations to deal with the nature of functions of an octonion variable. [9][10] In the case of quaternions, [11] it was shown that Maxwell's equations are reduced to the regularity condition for functions of a biquaternion variable. This made it possible to derive the nature of an electromagnetic field through the use of the theory of functions of a biquaternion variable [11].

Thus, an electromagnetic field is expressed by a regular function of a biquaternion variable and a regular function expresses an electromagnetic field if a proper initial condition is implemented. Therefore, the theory of functions is, in some way equivalent to the theory of electromagnetic fields.

As an extension from the quaternions to octonions, we see that the octonion wave equation proposed by Penney is equivalent to the regularity condition for function of an octonion variable. In the theory of functions of an octonion variable, the regular octonion functions which satisfy the regularity condition satisfy the octonion equation so that a regular function of an octonion variable represents an octonion wave function and the mathematical theory of functions becomes essentially equivallent to the physial theory of an octonion wave function if a proper correspondence between the two theories being established.

In this paper, we describe the theory of functions of an octonion variable in relation to the octonion wave equation and to derive the solution of the octonion wave equation using the theory of functions of a split octonion variable.

2. Octonions and theory of functions of an octonion variable and its extention to split octonions.

In recent papers,(9 10 11) the theory of functions of an octonion variable and those of a hypercomplex variable of Cayley-Dickson algebras and of Clifford algebra have been developed and some of the results have been published in this Bulletin. Therefore, we do not elaborate the theory of functions in this paper but we quote freely the results to our present purpose in this paper when we need them. However, we have to extend the theory to split (complex) octonions. The reason for this is as follows.

The norm of an octonion is a quadratic metric of positive definite

$$\sum_{a=0}^{7} x_a^2$$

The space corresponds to this metric is not Minkowski space but is a Euclid space. To obtain Minkowski space as a sub-space, we need to extend the octonions to split octonions, the units of which is e_α ($\alpha = 1, 2, 3, \cdots\cdots, 7$) which are related to the octonion units i_α ($\alpha = 1, 2, \cdots\cdots, 7$) by the following relations:

$$(1) \quad e_\alpha = i\,i_\alpha, \qquad i = \sqrt{-1}\,.$$

As has been done in the case of the theory of quaternion functions , the complexification of octonions is not a trivial matter in the case of the function theory as will be shown later. Because the split octonions as in the case of quaternions, are not a division algebra but contain zero divisors. This makes a fundamental change in the character of the theory of functions, though in the algebra zero divisors have to be added.

2—A) The algebra of split octonions.

The multiplication rules for octonion units i_α is given by

(2) $i_\alpha i_\beta = -\delta_{\alpha\beta} + \varepsilon_{\alpha\beta\gamma} i_\gamma, \; i_0 = 1.$

where $\varepsilon_{\alpha\beta\gamma}$ are the structure constants: totally antisymmetric with respect to $\alpha, \beta, \delta = (1, 2, 3, \cdots\cdots, 7)$:

(3) $\varepsilon_{\alpha\beta\gamma} = +1,$ for $\alpha\beta\gamma = 123, 145, 176, 572, 347, 365.$

Then, the multiplication rule for e_α is obtained from (2) as follows:

(4) $e_\alpha e_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma} e_\gamma, \; \overset{+}{e_\alpha} = -e_\alpha.$

There is no change in other algebraic rules except the introduction of zero divisors. For example: $X = x_0 + e_1 x_1$, $X^+ = x_0 - e_1 x_1$ are zero divisors when $x_0 = \pm x_1$, because X X^+ = $x_0^2 - x_1^2 = 0$. But neither X nor X^+ is zero.

2 − B) Functions of a complex (split) octoinion variable and the regularity condition.

We define a complex octonions $Z = \sum_{\alpha=0}^{7} e_\alpha z_\alpha$, $z_\alpha = x_\alpha + iy_\alpha$ and $X = \sum_{\alpha=0}^{7} e_\alpha x_\alpha \cdot x_\alpha \varepsilon R$, the real part of the complex octonion be designated as a "**physical octonion**". Then, we define the regularity condition for a function F (X) of a physical octonion variable X as follows (9 10) :

Define a differential octonion operator D by

(5) $D = \sum_{\alpha=0}^{7} e_\alpha (\partial/\partial x_\alpha)$

and let an octonion function F (X) be

(6) $F(X) = \sum_{\alpha=0}^{7} e_\alpha f_\alpha(x_0, \cdots, x_7),$

f_α are scalar function and $\varepsilon\, C^2$. Then, the function F(X) is left D regular (or right D regular) at X if and only if F(X) satisfies the following equation:

(7) DF (X) = 0, (F (X) D = 0),

where we define

$$DF(X) = \sum_{a,b=0}^{7} [(e_a e_b)\partial/\partial x_a] f_c(x_0, x_1, \cdots, x_7) = 0,$$

the equation is introduced in eq. (25') in ref. (1).

3. **Octonion wave equation.**

The octonion wave equation proposed by Penney is a special case of the regularity condition for functions of an octonion variable given by equation (7).

3 − A) **Derivation of the wave equation from the regularity condition.**

For the regularity condition we have discussed in ref. (10) and (11), the space is an 8 dimensional octonion space, while in the Penney's octonion wave equation, the basis space-time is the 3 + 1 space-time. We now derive the octonion wave equation from the regularity condition by confining the extra four dimensional subspace into the internal space so that the extra internal space variables no longer appear in the wave equation but appear, instead, through the internal symmetry or the internal quantum numbers.

Let the regularity condition (7) be written as:

$$(8)\quad \left(\partial_0+\sum_{k=1}^{3} e_k\partial_k+\sum_{a=4}^{7} e_a\partial_a\right)\Phi(x_0,\ \cdots,\ x_3,\ x_4,\ \cdots,\ x_7)=0,$$

or

$$(9)\quad (D_1+D_2)\Phi(X_1,\ X_2)=0. \qquad X_1=\sum_{\mu=0}^{3} e_\mu x_\mu,\ \ X_2=\sum_{a=4}^{7} e_a x_a.$$

We assume $\Phi(X_1,\ X_2)$ being a product function of $\Phi_1(X_1)$ and $\Phi_2(X_2)$: the one depends only on x_0 through x_3 and the other only on x_4 through x_7 : $\Phi=\Phi_2\Phi_1$. Also we assume that the function $\Phi_2(X_2)$ is a scalar function so that it commutes with all quantities: e_α, (α = 0, 1, ⋯⋯, 7). Then, we obtain by the method of separation of the variable, as follows:

$$(10)\quad (D_1+D_2)(\Phi_2\Phi_1)=(D_2\Phi_2)\Phi_1+\Phi_2(D_1\Phi_1)=0.$$

Multiplying $(\Phi_2^{-1}\Phi_1^{-1})=\Phi^{-1}$ from the right of the above equation and using Φ_2 commutes with all quantities, we find

$$(11)\quad (D_2\Phi_2)\Phi_2^{-1}=-(D_1\Phi_1)\Phi_1^{-1}.$$

Since the both sides of (11) depend on different variables so that it should be equal to a constant: let the constant be put equal to: $im=\sum_{a=4}^{7} ie_a m_a$. Because the left-hand side contains only $e_a(a$ = 4, 5, ⋯ 7) and not e_μ (μ = 0, 1, 2, 3), we can equate the left hand side of the above to be equal to $im=$

$\sum_{a=2}^{7} ie_a m_a$. Thus, we have

(12−1) $D_2\Phi_2 = -im\Phi_2$,

(12−2) $D_1\Phi_1 = im\Phi_1$.

The first equation which contains only internal coordinate variables (x_4, …… x_7) and determines the constants m_a is the eigenvalue equation in the internal space to determine the internal quantum numbers m_a. The second equation containning only the space-time (external) variables $x_0, \cdots, x_3$ is the equation used by penney.

3 − B) **Derivation of Proca's equation and its extension.**

We derive another form of an octonion wave equation which is called Proca's equation in the following.

Let us take a special case of eq. (7):

$$D = D_1 + D_2, \quad D_2 = ime_4, \quad D_1 = \sum_{\mu=0}^{3} e_\mu \partial_\mu, \tag{13}$$

$$\Phi = \sum_{\mu=0}^{3} e_\mu \phi_\mu + e_4 \phi_4$$

and the octonion wave equation be (in analogy with the quaternion wave equation of Maxwell's equations[11])

$$F = \overline{D}\Phi, \tag{14}$$

$$S = DF,$$

where Φ, F and S are the potential function, the field function and the source function, respectively, and are given by the real component functions as follows:

$$\Phi = \phi_0 + \underline{\phi} + ie_4\phi_4,$$

$$F = \underline{f} + i\underline{g} + ie_4(u_0 + \underline{u}) \tag{15}$$

$$S = i_0 + \underline{i} + ie_4(\underline{t} + i\underline{s})$$

where $\underline{\phi} = \sum_{k=1}^{3} e_k \phi_k$ is a space vacxtor.

The equations (15) are, given in the space-time vector form as follows:

$$f_0 = \partial_0\phi_0 - div\,\underline{\phi} + m\phi_4,$$

$$\underline{f} = \partial_0\underline{\phi} - grad\,\phi_0,$$

$$\text{(16)}\quad \begin{aligned} &\underline{g}=-\,curl\,\underline{\phi}\,,\\ &u_0=\partial_0\phi_4-m\phi_0\,,\\ &\underline{u}=-grad\,\phi_4-m\underline{\phi}\,,\end{aligned}$$

and

$$\text{(17)}\quad \begin{aligned} &i_0=\partial_0 f_0+div\underline{f}-mu_0\,,\\ &\underline{i}=\partial_0\underline{f_0}+grad f_0-curl\,\underline{g}-m\underline{u}\,,\\ &\underline{s}=m\underline{g}+curl\,\underline{g}\,,\\ &\underline{t}=\partial_0\underline{u}-grad\,u_0+m\underline{f}\,,\\ &0=div\,\underline{g},\\ &0=\partial_0\underline{g}+curl\,\underline{f}\,,\\ &0=\partial_0 u_0-div\,\underline{u}+mf_0\,,\end{aligned}$$

The equations (14) and (15) are the Proca's equation(14) in the octonion form(1). The mass of the field Φ can be found as follows. Multiply D^{+} from the left

$$F=\overline{D}\Phi$$

of the second equation of (14), the equation satisfies the following:

$$\text{(19)}\quad \overline{D}D\Phi=(\Box-m^2)\,\Phi=S\,,$$

when S = 0, each components of Φ satisfy the same equation as

$$\text{(20)}\quad (\Box-m^2)\,\phi_\mu=0.$$

The equation (19) shows that m is the mass of the field. This form of Proca's equation has been introduced in (13).

The equation (14) can be generalized by extending the scalar mass m to a four component vector mass: $m=\sum_{a=4}^{7} e_a m_a$ as described in the following.

3 − C) **Extention of the Proca's equation to include vector mass.**

The equation (13) can be generalized by extending the scalar mass m to a four dimensional vector mass:

$$\text{(20)}\quad m=\sum_{a=4}^{7} e_a m_a.$$

For this purpose, we extend the differential operator D, Φ, F, and S to a general case of octonions which include the units e_5, e_6, e_7. Then, the equations can be written as follows: Let

(21) $D=\sum_{a=0}^{7} e_a\partial_a, \quad \Phi=\sum_{a=0}^{7} e_a\Phi_a,$

$F=\sum_{a=0}^{7} e_a(f_a+ig_a), \quad S=\sum_{a=0}^{7} e_a(t_a+iu_a),$

and the field equations be

(22) $\bar{D}\Phi=F,$

$DF=S,$

we obtain similar equations as (16) and (17) which are Proca's equation extended to a general case of octonions.

4. Solutions of the octonion wave equations.

As described in the above sections, the octonion wave equations are closely related to the regularity condition for functions of an octonion variable: special cases of the regularity condition. Therefore, the regular functions which satisfy the regularity condition, if certain constraints are imposed, will be the solutions of the octonion wave equations.

In the theory of functions of an octonion variable, several methods of obtainning regular functions are introduced.[10] Thus, we apply these methods together with the ordinary method to obtain the solutions of the octonion wave equations as described in the following.

4 − A) A free field case.

(1) Eigenvalue equation

Equation (12 − 2) can be written in the Hamiltonian form:

(23) $i(\partial/\partial x_0)\Phi_1=H\Phi, \quad H=i[\sum_{k=1}^{3} e_k\partial_k+\sum_{a=4}^{7} e_a m_a]$

where H is the Hamiltonian of the field.
Now we seek a free field solution of the form exp $[-i\sum_{b=0}^{7} p_b x_b]$.
Let us assume that Φ_1 be given as follows:

(24) $\Phi_1=\sum_{a=0}^{7}[u_a e_a]\,exp[-i\sum_{a=0}^{7} p_a x_a - i\sum_{c=4}^{7} m_c x_c]$

where u_a are octonion constants. Inserting Φ_1 of (24) in (23), we find p_0

should satisfy the following eigenvalue equation:

(25) $\Delta = |k_{ab} - p_0 \delta_{ab}| = 0,$

where p_0 is the eigenvalue, $\Delta = |k_{ab} - p_0 \delta_{ab}|$ is defined by the following determinant

(26)
$$\Delta = \begin{vmatrix} p_0 & p_1 & p_2 & p_3 & m_4 & m_5 & m_6 & m_7 \\ p_1 & p_0 & ip_3 & ip_2 & -im_5 & im_4 & im_7 & -im_6 \\ p_2 & ip_3 & p_0 & -ip_1 & -im_6 & -im_7 & im_4 & im_5 \\ p_3 & -ip_2 & ip_1 & p_0 & -im_7 & im_6 & -im_5 & im_4 \\ m_4 & im_5 & im_6 & im_7 & p_0 & -ip_1 & -ip_2 & -ip_3 \\ m_5 & -im_4 & im_7 & -im_6 & ip_1 & p_0 & ip_3 & -ip_2 \\ m_6 & -im_7 & -im_4 & im_5 & ip_2 & -ip_3 & p_0 & ip_1 \\ m_7 & im_6 & -im_5 & -im_4 & ip_3 & ip_2 & -ip_1 & p_0 \end{vmatrix}$$

Now, we can solve the equation (25). Multiplying the following determinant to Δ : i.e. change all the $p_k (k = 0, 1, 2, \cdots\cdots, 7)$ to $-p_k$ in the first column of Δ and also do the same to the first row, we obtain another determinant: $\Delta' = |k_{ab} - p_0 \delta_{ab}|$. Then, multiply this determinant Δ' from the right of Δ of (26), we find

$$\Delta\Delta' = (p_0^2 - p_1^2 - p_2^2 - p_3^2 - \sum_{a=4}^{7} m_a^2)^8$$

Since $\Delta = \Delta'$, we have

(27) $|k_{ab} - p_0 \delta_{ab}| = \pm \left[p_0^2 - \sum_{k=1}^{3} p_k^2 - \sum_{a=4}^{7} m_a^2 \right]^4 1 = 0.$

Thus, we have eight eigenvalues for the energy p_0:

(28) $p_0 = E_0 = \sqrt{\sum_{k=1}^{3} p_k^2 + \sum_{a=4}^{7} m_a^2},$ four roots,

$p_0 = -E_0 = -\sqrt{\sum_{k=1}^{3} p_k^2 + \sum_{a=4}^{7} m_a^2},$ four roots,

These eightfold eigenvalues correspond to the positive and negative values of the energy, spin and isospin, respectively.

4 − B) Wave functions derived from the regular functions of an octonion variable

Now, we derive the solution of the wave equation from the regular functions of an octonion variable using the Fourier integral representation.

From the equation (15) in ref 12, we have a general solution of the equation of the regularity condition (7) as follows:

(29) $$\Phi(X)=\frac{1}{(2\pi^4)^7}\int_{-\infty}^{+\infty}\cdots\int exp[\pm i(\underline{t}x_0+(\underline{t}\cdot\underline{s}))]A(\underline{t})d^7\underline{t},$$

where $\underline{t}$ is a sum of the vector $\underline{t}=\sum_{k=1}^{3}t_ke_k$ in the external space $(x_0,\cdots,x_3)$ and the four vector $\boldsymbol{t}=\sum_{a=4}^{7}t_ae_a$ in the internal space $(x_4,\cdots,x_7)$.(*)

(*) We put underbar _ for a space 3 - vector in the Minkowski space and we write a 4 vector in the internal space by a bold letter. An octonion is written by a sum of two vectors $\underline{t}$ and t and a scalar t_0 as $T=t_0+\underline{t}+\boldsymbol{t}=t_0+\underline{t}$ which are specific notations used in this paper only.

From eq. (29), $A(\underline{t})$ is determined by the initial condition:

(30) $$\Phi(X)|_{x_0=0}=\Phi(\underline{x})$$

as follows. Putting in (29) $x_0=0$, we have

(31) $$\Phi(\underline{x})=(2\pi)^{-7}\int_{-\infty}^{+\infty}\cdots\int exp[+(\underline{t}.\underline{x})]A(\underline{t})d^7\underline{t},$$

where $A(\underline{t})$ satisfies certain constraints.

(32) $$\int_{-\infty}^{+\infty}\cdots\int|N(A(\underline{t}))|\,d^7\underline{t}=\text{ finite}$$

Since the right hand side of the above equation, A $(\underline{t})$ is the Fourier transform of $\Phi(\underline{x})$. Inverting the integral we find

(33) $$A(\underline{t})=\int\cdots\int exp[\mp(\underline{\tau}.\ \underline{t})]\Phi(\underline{x})d^7\underline{\tau}.$$

Now, take $A(\underline{t})$ a specific case as:

(34) $$A(\underline{t})=(2\pi)^4A_1(\underline{t})\prod_{a=4}^{7}\delta(t_a-m_a),$$

where δ (x) is the δ-function. Then, we find from (31):

$$(35)\quad \Phi(X)=(2\pi)^{-3}\int_{-\infty}^{+\infty}\cdots\int exp[+i(\underline{t}x_0+\boldsymbol{m}x_0)]exp[+i((\underline{t}\cdot\underline{x})+\boldsymbol{m}\cdot\boldsymbol{x})]A(\underline{t})d^3\underline{t}.$$

This is a wave function which satisfies the octonion wave equation (12－1). In the above, since $\boldsymbol{m}$ and t do not commute:

$$(36)\quad exp[i(\underline{t}x_0+\boldsymbol{m}x_0)]\neq exp[i\underline{t}x_0]exp[i\boldsymbol{m}x_0],$$

we cannot write the function as a product of two exponential functions, one depends only on the external variables and the other on the internal variables.

The Fourier integral form of the solution is suitable for the quantization of the field function (35). The field quantization of the solution of the field (35) will be dealt with in a future paper.

4 － C) **The solution of a generalized Proca's equation when there is a source.**

The equation (14) can be solved using the residues theorem given in section 6 in ref. (10). However, in our case, the space contains Minkowski space-time so that we have to extend the residue theorem to the split octonion variable.

Since split octonions contain zero divisors, the singularity is spread over a seven dimensional hypersurface in the space $(x_0, x_1, \cdots, x_7)$ where x_μ are all real scalar quantites but the norm of a physical octonion X is given by:

$$(37)\quad N(X)=XX^+=x_0^2-x_1^2-\cdots-x_7^2,$$

So that when $x_0 = \pm\sqrt{x_1^2+\cdots+x_7^2}$, , the product of X and its conjugate X is a zero divisor.

Thus, when we extend the residues theorem from an octonion variable which is given in ref (10) as

$$(38)\quad (48\pi^4)^{-1}\int[(F(X)dX)(D\square(Z-X)^{-1})]=F(Z),$$

to the complex octonion variable, we need to change the surface of integration S^7 from a closed hypersurface to a surface which encircles the singularity surface of the zero divisor $N(Z-X)^{-1}$ so that it extends into a split octonion space.

The procedure is quite similar to the case of extending the residues theorem of a quaternion variable to that of a complex quaternion variable as illustrated in ref. (11). But the nonassociative nature of octonions makes the derivation of a field function from residues theorem is a bit complex. Thus, the extenion of the theory of functions of an octonion variable to that of a complex (split) octonion variable is needed so that the theory based on the residues theorem will be dealt with in another paper and will not be ellaborate in this paper.

References:

1) K. Imaeda: Prog. Theor. Phys., **2**, 133 (1950),; Memoirs of the Faculty of Liberal Arts and Education. No. 2, 111 (1951); Report-Fundamental Physics Laboratory, **1** (1983).

2) R. Penney: Amer, Journ. Phys., **36** 1651 (1968); Nuovo Cimento, **3** × 871 (1971).

3) M. Gunaydin and F. Gürsey: J. Math, Phys. **14** 1651 (1973).

4) F. Gürsey: "To **Fulxfillx** a Vision, Jerusalem Einstein Centennial Symposium on Gauge Theories and Unification of Physical Forces" Ed. Y. Ne'eman, Addison-Wesley Pub. Co. Inc. 1981, p22−38.

5) F. Englart: Phys, Lett. **119B** 339 (1982).

6) M. Rooman: Nuclear Physics, **B236** 501 (1984), F. Gürsey and C-H. Tze: Phys. Lett., **172B** 191 (1983).

7) R. Dündarer, F. Gürsey and C-H. Tze: Nuclear Phys. **B266**, 440 (1986).

8) G. C. Joshi: Lett. Nuovo Cimento: **44** 449 (1985).

9) P. Dentoni and M. Sce: Rend. Sem. Mat. Padova. **50** 251 (1973).

10) K. Imaeda and Mari Imaeda: Bulletin Okayama Univ. Science; **19A** 93 (1984).

11) K. Imaeda: Nuovo Cimento **22B** 138 (1976).
Mari Imaeda: „ Clifford Algebras and Their Applications in mathematical Physics, Ed. by J.S.R.Chisholm and A.K.Common. D.Reidel Pub. Co. 1986.pp565-572

12) K. Imaeda and Mari Imaeda: Bulletin Okayama Univ. Science, **A133** (1985).

13) K. Imaeda: Prog. Theor. Phys., **2** 133 (1950).

14) H. Yukawa, S. akata, and M, Taketani: Proc. Phys. - Math. Soc. **20**, 319 (1938).