Examples of Localized Solutions of Non-Linear Klein-Gordon Equations

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ABSTRACT

Some examples of localized solutions of non-linear Klein-Gordon equations in $1+1$ and $3+1$ dimensional space-time are presented with their generalized virial relations corresponding to Derrick's condition for the existence of soliton-like solution.

1. INTRODUCTION

In many branches of Physics, a great interest in various types of non-linear effects is rapidly growing. Of particular interest in the elementary particle physics is such attempts to find any non-linear field equation which admits the existence of solitons, i.e. stable and localized solutions. Many examples of such non-linear equations and their solutions in the $1+1$ dimensional space-time are already known [1), 2)]. As for the $3+1$ dimensioal space-time, unfortunately, we know only a few examples which are summarized as follows.

Droplet-like stationary solutions of non-linear Klein-Gordon ($K-G$) equations for a complex scalar field as well as droplet-like spinor solutions of non-linear Dirac equations have been found by J. Werle [3), 4)].

I. Bialynicki-Birula and J. Mycielski have presented Gaussian shaped solutions of complex scalar fields for another class of non-linear $K-G$ equations [5)].

Abgebraic solutions of non-linear $K-G$ or Dirac equations in the $3+1$ dimensions are given by M. Umezawa recently[6].

In this paper we shall show that also another classes of non-linear $K-G$ equations in $1+1$ and $3+1$ dimensional space-time provide the localized hyperbolic secant (sech)-type solutions.

We know several weak necessary conditions for the existence of soliton-like solutions. In the case of static real scalar field, that is Derrick's condition[7] corresponding to a virial relation between the kinetic energy part and the potential energy part of the field.
In the case of complex scalar field, there exist another different virial conditions. In section 2 we shall give these virial relations. In section 3 and 4, we shall present a class of static and stationary sech-type solutions of non-linear $K-G$ equation in the $1+1$ and $3+1$ dimensions.
Section 5 is devoted to the discussion in which our sech-type solutions are compared with the soliton-like solutions summarized above.

2. VIRIAL RELATIONS

J. Werle investigated the global method, made use of variable initial conditions, in order to find the conditions for types of non-linearities which allow only non-dissipative, confined solutions[8]. In this paper we follow his method.

(i) Real scalar field case

Consider the real scalar field ϕ in the $1+D$ dimensions with the metric $(1, -1, -1, \cdots\cdots, -1)$. The Lagrangian density is given by

$$\mathscr{L}=\frac{1}{2}\partial_\nu\phi\partial^\nu\phi-U(\kappa), \tag{1}$$

where $U(\kappa)$ is some non-linear function of

$$\kappa=\phi^2. \tag{2}$$

The corresponding non-linear Klein-Gordon equation is

$$\left(\Box+2\frac{dU}{d\kappa}\right)\phi=0, \tag{3}$$

where the $1+D$ dimensional D'Alembertian is

$$\Box=\frac{\partial^2}{\partial t^2}-(\vec{\nabla})^2 \tag{4}$$

and $\vec{\nabla}$ is the D dimensional gradient.

For a given $U(\kappa)$, a solution of (3) is determined by the initial values of the field at some fixed time t_0;

$$\varphi(\vec{x})=\phi(\vec{x},\ t_0), \qquad \varphi_0(\vec{x})=\left.\frac{\partial\phi(\vec{x},\ t)}{\partial t}\right|_{t=t_0}. \tag{5}$$

The energy-momentum density tensor is defined by

$$t_{\alpha\beta}=\partial_\alpha\phi\partial_\beta\phi-\delta_{\alpha\beta}\mathscr{L} \tag{6}$$

which obeys the equation of continuity $\partial_\beta t^{\alpha\beta}=0$.

Hence the energy and the k-th component of momentum can be now expressed in terms of the initial values (5);

$$\begin{aligned} E&=\int d^Dx\,t_{00}\\ &=\int d^Dx\left\{\frac{1}{2}(\varphi_0)^2+\frac{1}{2}(\vec{\nabla}\varphi)^2+U(\varphi^2)\right\}, \end{aligned} \tag{7}$$

$$\begin{aligned} P_k&=\int d^Dx\,t_{0k}\\ &=\int d^Dx\,\varphi_0\frac{\partial\varphi}{\partial x_k}. \end{aligned} \tag{8}$$

We introduce the following class of finite distortions of the initial values.

$$\begin{aligned} &\varphi(\vec{x})\to\tilde{\varphi}(\vec{x})=c\varphi(\vec{x}/a),\\ &\varphi_0(\vec{x})\to\tilde{\varphi}_0(\vec{x})=bc\varphi_0(\vec{x}/a), \end{aligned} \tag{9}$$

where a, b and c are positive parameters.

We require that the momentum of (8) should be invariant under the transformation (9).

First, the same valueness of (8) gives

$$a^{D-1}bc^2=1. \tag{10}$$

Thus, one can eliminate c in the expression of transformed energy. For any given $\varphi(\vec{x})$ and $\varphi_0(\vec{x})$ the energy functional becomes a definite function of a and b alone;

$$E(a,\ b)=a^{-1}b^{-1}T_1+abT_2+a^D\int d^Dx\,U\left[a^{1-D}b^{-1}\varphi^2\right], \tag{11}$$

where

$$T_1=\int d^Dx\frac{1}{2}(\vec{\nabla}\varphi)^2\geqq 0, \tag{12}$$

$$T_2=\int d^Dx\frac{1}{2}\varphi_0^2\geqq 0. \tag{13}$$

According to Derrick's procedure [7], $\partial E(a,\ b)/\partial a$ and $\partial E(a,\ b)/\partial b$ are zero at $a=b=1$. From (11), we obtain

$$\left.\frac{\partial E(a,\ b)}{\partial a}\right|_{a=b=1}=-T_1+T_2+DV_1+(1-D)V_2=0 \tag{14}$$

and

$$\left.\frac{\partial E(a,\ b)}{\partial b}\right|_{a=b=1}=-T_1+T_2-V_2=0, \tag{15}$$

where

$$V_1=\int d^Dx\, U(\varphi^2) \tag{16}$$

and

$$V_2=\int d^Dx\, \varphi^2\frac{dU(\varphi^2)}{d(\varphi^2)} \quad . \tag{17}$$

We eliminate V_2 from (14) and (15), then we get

$$(2-D)\ (T_2-T_1)+DV_1=0. \tag{18}$$

When the field is static, eq.(18) becomes the well-known Derrick's relation.

(ii) Complex scalar field case

In this case, the Lagrangian density is defined by

$$\mathscr{L}=\partial_\nu\phi^*\partial^\nu\phi-U(\kappa), \tag{19}$$

where $U(k)$ is a certain non-linear function of

$$\kappa=\phi^*\phi \tag{20}$$

and * denotes complex conjugate.

From (19), we have the following non-linear $K-G$ equation ;

$$\left(\Box+\frac{dU}{d\kappa}\right)\phi=0. \tag{21}$$

The energy and charge functionals can be now expressed in terms of (5) ;

$$E=\int d^Dx\ \{|\varphi_0|^2+|\vec{\nabla}\varphi|^2+U(|\varphi|^2)\} \tag{22}$$

$$Q=\frac{i}{2}\int d^Dx\ \{\varphi^*\varphi_0-\varphi_0{}^*\varphi\}. \tag{23}$$

The invariancy of Q value under the transformation corresponding to (9) gives the relation ;

$$a^Dbc^2=1.$$

Following the procedure in the case of (i), we obtain the virial relations corresponding to (14) and (15) ;

$$-2\tilde{T}_1+D(\tilde{V}_1-\tilde{V}_2)=0 \tag{24}$$

and

$$-\tilde{T}_1+\tilde{T}_2-\tilde{V}_2=0, \tag{25}$$

where

$$\tilde{T}_1 = \int d^D x |\vec{\nabla} \varphi|^2 \geqq 0, \tag{26}$$

$$\tilde{T}_2 = \int d^D x |\varphi_0|^2 \geqq 0, \tag{27}$$

$$\tilde{V}_1 = \int d^D x \, U(|\varphi|^2), \tag{28}$$

and

$$\tilde{V}_2 = \int d^D x |\varphi|^2 \frac{dU(|\varphi|^2)}{d(|\varphi|^2)}. \tag{29}$$

From (24) and (25), eliminating $\tilde{V}_2$ we get

$$(D-2)\tilde{T}_1 - D(\tilde{T}_2 - \tilde{V}_1) = 0. \tag{30}$$

3. HYPERBOLIC SECANT TYPE SOLUTIONS FOR NON-LINEAR *K-G* EQUATIONS IN THE 1+1 DIMENSION

(i) Static case

(Example 1)

In the Lagrangian density (1), we assume the following form of non-linear potential (see Fig. 1) ;

$$U(\kappa) = n^2 \alpha^2 \kappa \left(1 - \kappa^{\frac{1}{n}}\right), \tag{31}$$

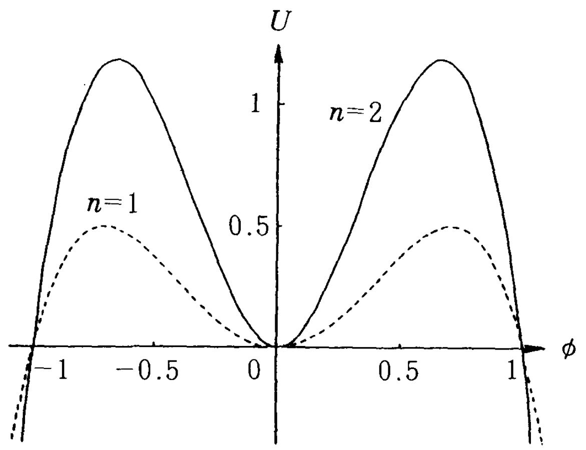


Fig. 1. The potential $U(\phi)$ of sech(x)-type solution in the static $D=1$ case.
{ The dashed curve for $n=1$ and $\alpha^2=2$,
The solid curve for $n=2$ and $\alpha^2=2$.

where n is integer with $n \geqq 1$ and α is arbitrary constant which has dimension of $[L^{-1}]$.

If the real scalar field ϕ is independent of time, that is a static case, the non-linear K—G equation (3) becomes

$$\frac{d^2\phi(x)}{dx^2} = 2n\alpha^2 \{n-(n+1)(\phi(x))^{2/n}\}\phi(x). \tag{32}$$

in the one spatial dimension.

It is well-known that eq. (32) has the solution [9];

$$\phi(x) = \mathrm{sech}^n(\sqrt{2}\,\alpha x). \tag{33}$$

This is a localized solution, as shown in Fig. 2.

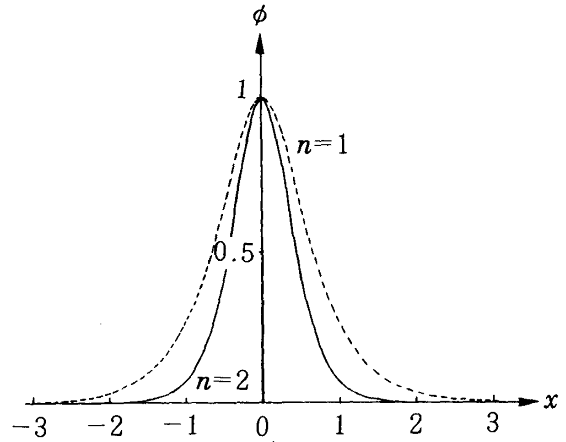


Fig. 2. The sech(x)-type solution for $\alpha^2=2$ in the static $D=1$ case.
The dashed curve for $n=1$,
The solid curve for $n=2$.

From eq.(7), the field energy is

$$E = \int_{-\infty}^{\infty} dx \left\{\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + U(\phi^2)\right\}$$
$$= T_1 + V_1. \tag{34}$$

From eq.(31) and (33), we can show

$$T_1 = V_1. \tag{35}$$

Note that eq.(35) coincides with the Derrick's relation for $D=1$.

We can calculate the field energy of (33). We have

$$E = 2T_1$$
$$= \int_{-\infty}^{\infty} dx \left(\frac{d\phi}{dx}\right)^2$$
$$= 2n^2\alpha^2 \int_{-\infty}^{\infty} dx\, \mathrm{sech}^{2n}(\sqrt{2}\,\alpha x)\tanh^2(\sqrt{2}\,\alpha x)$$
$$= 2\sqrt{2}\,\alpha \frac{n^2(2n-2)!!}{(2n+1)!!}. \tag{36}$$

We see that the energy is finite. Therefore, the solution (33) of the non-linear $K-G$ equation (32) gives a soliton.

(Example 2)

When the non-linear potential in the Lagrangian density is given by

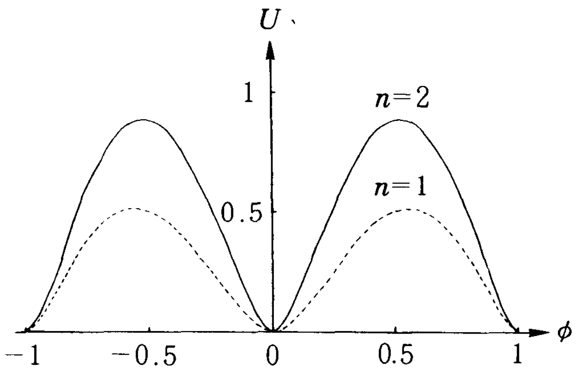


Fig. 3. The potential $U(\phi)$ of sech(x^2)-type solution for $\alpha^2=2$ in the static $D=1$, case.
{ The dashed curve for $n=1$,
{ The solid curve for $n=2$.

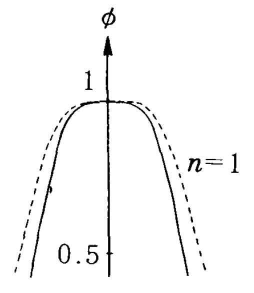
$$U(\kappa)=n^2\alpha^2\kappa(1-\kappa^{1/n})\ln\frac{1+\sqrt{1-\kappa^{1/n}}}{\kappa^{1/2n}} \tag{37}$$

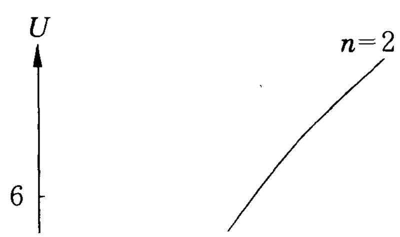
(see Fig. 3), we have the non-linear $K-G$ equation

$$\frac{d^2\phi(x)}{dx^2}=n\alpha^2\Big[2\{n-(n+1)(\phi(x))^{2/n}\}\ln\frac{1+\sqrt{1-(\phi(x))^{2/n}}}{(\phi(x))^{1/n}} - \sqrt{1-(\phi(x))^{2/n}}\Big]\phi(x). \tag{38}$$

The solution of (38) is

$$\phi(x)=\mathrm{sech}^n(\alpha^2x^2/2) \tag{39}$$





which is localized solution as shown in Fig. 4.

We can easily check that the virial relation for $D=1$ is satisfied by (37) and (39).

The field energy of this solution is

$$E=2\int_{-\infty}^{\infty} dx\,\varepsilon(x), \tag{40}$$

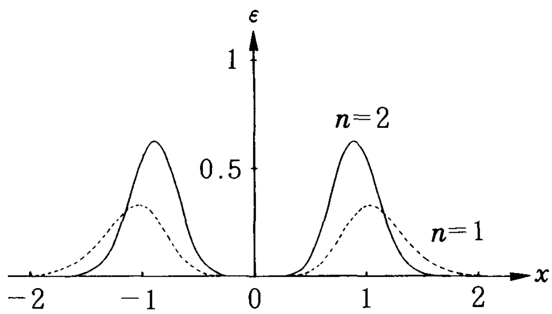


Fig. 5. $\varepsilon(x)=n^2\alpha^2\mathrm{sech}^{2n}(\alpha^2x^2/2)\tanh^2(\alpha^2x^2/2)\,1n\dfrac{1+\tanh(\alpha^2x^2/2)}{\mathrm{sech}(\alpha^2x^2/2)}$

{ The dashed curve for $n=1$ and $\alpha^2=2$,
The solid curve for $n=2$ and $\alpha^2=2$.

where the energy density

$$\varepsilon(x)=U(\kappa)$$
$$=n^2\alpha^2\mathrm{sech}^{2n}(\alpha^2x^2/2)\tanh^2(\alpha^2x^2/2)\times 1n\frac{1+\tanh(\alpha^2x^2/2)}{\mathrm{sech}(\alpha^2x^2/2)} \tag{41}$$

which is shown in Fig. 5.

For $n=1$ and $n=2$ case, the numerical integration of (40) gives the finite value respectively

$$\begin{cases} E\simeq 1.41 & \text{for } n=1 \text{ and } \alpha^2=2, \\ E\simeq 1.87 & \text{for } n=2 \text{ and } \alpha^2=2. \end{cases}$$

(ii) Stationary case

(Example 3)

We choose the non-linear potential density in the Lagrangian density (19), as follows (see Fig. 6)

$$U(\kappa)=n^2\alpha^2\kappa(2-\kappa^{1/n}), \tag{42}$$

where κ is defined by (20).

The non-linear K—G equation (21) is

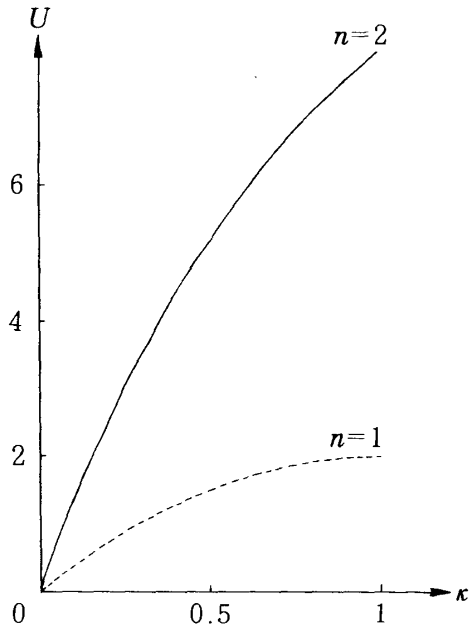


Fig. 6. The potential $U(\kappa)$ of sech(x)-type solution for $\alpha^2=2$ in the stationary $D=1$ case.
{ The dashed curve for $n=1$,
{ The solid curve for $n=2$.

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+n\alpha^2\{2n-(n+1)|\phi|^{2/n}\}\right]\phi=0. \tag{43}$$

A stationary solution of eq. (43) is given by

$$\phi(x,\ t)=\exp(-in\alpha t)\,\mathrm{sech}^n(\alpha x). \tag{44}$$

With eq. (42) and (44), we calculate (26), (27), (28) and (29) for $D=1$, then we get

$$\widetilde{T}_1=\alpha n\frac{(2n)!!}{(2n+1)!!}, \tag{45}$$

$$\widetilde{T}_2=\alpha n\frac{(2n)!!}{(2n-1)!!}, \tag{46}$$

$$\widetilde{V}_1=\alpha n\frac{(2n+2)!!}{(2n+1)!!} \tag{47}$$

and

$$\widetilde{V}_2=2\alpha n^2\frac{(2n)!!}{(2n+1)!!}. \tag{48}$$

We can show that virial relations (24), (25) and (30) for $D=1$ are satisfied. Hence, we have

$$\widetilde{T}_1+\widetilde{T}_2-\widetilde{V}_1=0. \tag{49}$$

The field energy in terms of (47) and (49) is

$$E=\widetilde{T}_1+\widetilde{T}_2+\widetilde{V}_1$$
$$=2\widetilde{V}_1$$
$$=2\alpha n\frac{(2n+2)!!}{(2n+1)!!}. \tag{50}$$

(Example 4)

For $U(\kappa)=n^2\alpha^2\kappa\left\{1+2(1-\kappa^{1/n})\ln\frac{1+\sqrt{1-\kappa^{1/n}}}{\kappa^{1/2n}}\right\}$ (51)

in the Lagrangian density (19) (see Fig. 7),

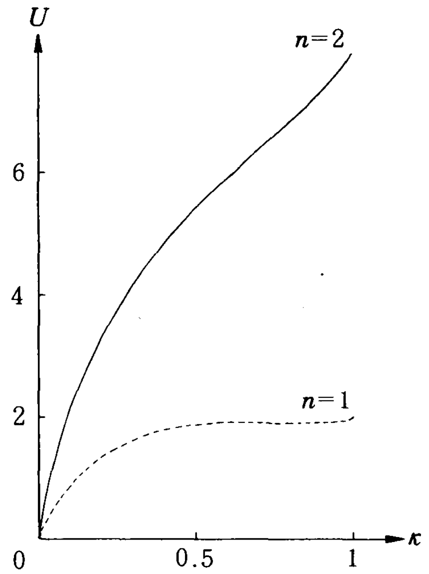


Fig. 7. The potential $U(\kappa)$ of sech(x^2)-type solution for $\alpha^2=2$ in the stationary $D=1$ case.
The dashed curve for $n=1$,
The solid curve for $n=2$.

the non-linear K—G equation is

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+n\alpha^2\left[n+2\{n-(n+1)|\phi|^{2/n}\}\ln\frac{1+\sqrt{1-|\phi|^{2/n}}}{|\phi|^{1/n}}\right.\right.$$
$$\left.\left.-\sqrt{1-|\phi|^{2/n}}\right]\right]\phi=0. \tag{52}$$

The stationary solution of (52) is

$$\phi(x,\ t)=\exp(-in\alpha t)\,\mathrm{sech}^n(\alpha^2x^2/2). \tag{53}$$

We can easily verify that the solution (53) satisfies the virial relation (49) and the field energy is

$$E=2\widetilde{V}_1=2\int_{-\infty}^{\infty}dx\,\varepsilon(x), \tag{54}$$

where $\varepsilon(x)$ is the energy density, for this example

$$\varepsilon(x)=n^2\alpha^2\,\mathrm{sech}^{2n}(\alpha^2x^2/2)$$
$$\times\left[1+2\tanh^2(\alpha^2x^2/2)\ln\frac{1+\tanh(\alpha^2x^2/2)}{\mathrm{sech}(\alpha^2x^2/2)}\right] \tag{55}$$

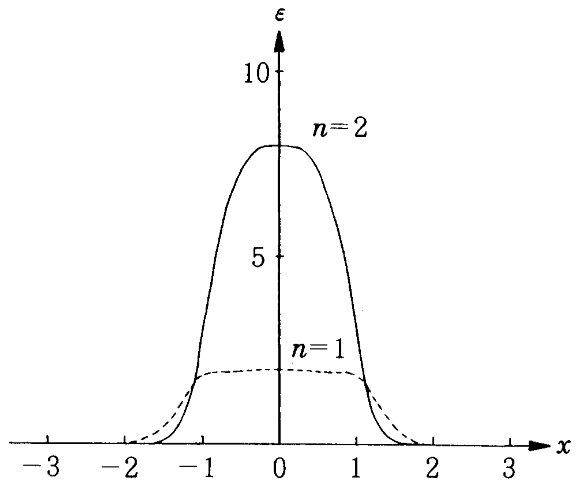


Fig. 8. $\varepsilon(x)=n^2\alpha^2\mathrm{sech}^{2n}(\alpha^2x^2/2)$
$\left[1+2\tanh^2(\alpha^2x^2/2)\ln\frac{1+\tanh(\alpha^2x^2/2)}{\mathrm{sech}(\alpha^2x^2/2)}\right]$

{ The dashed curve for $n=1$ and $\alpha^2=2$,
The solid curve for $n=2$ and $\alpha^2=2$.

which is shown in Fig. 8.

The energy for $n=1$ and 2 is evaluated by

$$\begin{cases} E\simeq 10.4 & \text{for } n=1 \text{ and } \alpha^2=2, \\ E\simeq 28.8 & \text{for } n=2 \text{ and } \alpha^2=2. \end{cases}$$

4. HYPERBOLIC SECANT TYPE SOLUTIONS FOR NON-LINEAR $K-G$ EQUATIONS IN THE 3+1 DIMENSIONS

(i) Static case

(Example 5)

Consider the Lagrangian density (1) with

$$U(\kappa)=n^2\alpha^4\Big[\kappa(1-\kappa^{1/n})\ln\frac{1+\sqrt{1-\kappa^{1/n}}}{\kappa^{1/2n}} +2\{A(\kappa)-B(\kappa)-\frac{(2n-2)!!}{(2n+1)!!}\}\Big] \tag{56}$$

Where κ is dimensionless form of

$$\kappa=\alpha^{-2}\phi^2, \tag{57}$$

$A(\kappa)$ and $B(\kappa)$ are

$$A(\kappa)=\frac{(2n-2)!!}{(2n-1)!!}\sqrt{1-\kappa^{1/n}}\sum_{m=0}^{n-1}\frac{(2n-2m-3)!!}{(2n-2m-2)!!}\kappa^{(n-m-1)/n} \tag{58}$$

and

$$B(\kappa)=\frac{(2n)!!}{(2n+1)!!}\sqrt{1-\kappa^{1/n}}\sum_{m=0}^{n}\frac{(2n-2m-1)!!}{(2n-2m)!!}\kappa^{(n-m)/n}. \tag{59}$$

Note that

$$2\{B(\kappa)-A(\kappa)\}=\frac{1}{n}\int\sqrt{1-\kappa^{1/n}}\,d\kappa. \tag{60}$$

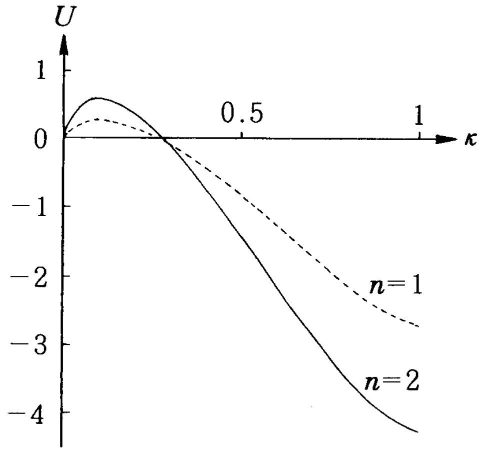


Fig. 9. The potential $U(\kappa)$ of sech(r^2)-type solution for $\alpha^2=2$ in the static $D=3$ case.
{ The dashed curve for $n=1$,
The solid curve for $n=2$.

The figure of this $U(k)$ is shown in Fig. 9.

The field equation of (3) is

$$\left[\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2+2\alpha^{-2}\frac{dU}{d\kappa}\right]\phi=0. \tag{61}$$

Here, we are looking for solutions which are static and spherical symmetric forms. Hence, eq. (61) is reduced to

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\phi(r)}{dr}\right]=n\alpha^2\Big[2\{n-(n+1)(\alpha^{-1}\phi)^{2/n}\}\times\ln\frac{1+\sqrt{1-(\alpha^{-1}\phi)^{2/n}}}{(\alpha^{-1}\phi)^{1/n}}-3\sqrt{1-(\alpha^{-1}\phi)^{2/n}}\Big]\phi(r). \tag{62}$$

We have the localized solution of (62);

$$\phi(r)=\alpha\,\mathrm{sech}^n(\alpha^2r^2/2). \tag{63}$$

The field energy is

$$E=T_1+V_1=4\pi\int_0^\infty dr\,t_1(r)+4\pi\int_0^\infty dr\,u_1(r)=4\pi\int_0^\infty dr\,\varepsilon(r) \tag{64}$$

where the energy density is $\varepsilon(r)=t_1(r)+u_1(r)$ with

$$t_1(r)=\frac{1}{2}r^2\left(\frac{d\phi}{dr}\right)^2=\frac{n^2\alpha^6}{2}r^4\tanh^2(\alpha^2r^2/2)\,\mathrm{sech}^{2n}(\alpha^2r^2/2) \tag{65}$$

and

$$u_1(r) = r^2 U(\kappa). \tag{66}$$

The Derrick's relation (18) for $D=3$ is expressed in terms of (65) and (66);

$$T_1 + 3V_1 = 4\pi \int_0^\infty dr \{t_1(r) + 3u_1(r)\}$$
$$\equiv 4\pi \int_0^\infty dr\omega(r) = 0. \tag{67}$$

The figure of the function $\omega(r) = t_1 + 3u_1$ is shown in Fig. 10.

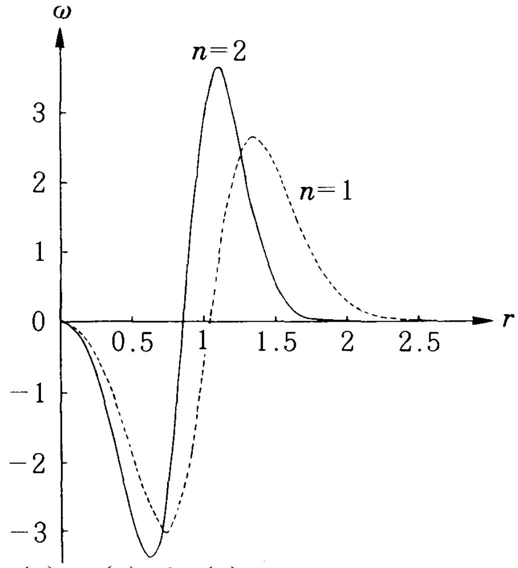


Fig. 10. $\omega(r) = t_1(r) + 3u_1(r)$

The dashed curve for $n=1$ and $\alpha^2=2$,
The solid curve for $n=2$ and $\alpha^2=2$.

By numerical integration, we find that eq. (67) is established. Hence, we have

$$E = \frac{2}{3} T_1$$
$$= \frac{8\pi}{3} \int_0^\infty dr t_1(r). \tag{68}$$

For $n=1$ and 2 case, the numerical integration gives

$$\begin{cases} E \simeq 8.01 & \text{for } n=1 \text{ and } \alpha^2=2, \\ E \simeq 7.17 & \text{for } n=2 \text{ and } \alpha^2=2. \end{cases}$$

(ii) Stationary case

(Example 6)

In the Lagrangian density (19), we introduce the non-linear potential (see Fig. 11)

$$U(\kappa)=n^2\alpha^4\Big[\kappa\{1+2(1-\kappa^{1/n})\ \ln\frac{1+\sqrt{1-\kappa^{1/n}}}{\kappa^{1/2n}}\}$$
$$+4\{A(\kappa)-B(\kappa)-\frac{(2n-2)!!}{(2n+1)!!}\}\Big], \tag{69}$$

where $A(\kappa)$ and $B(\kappa)$ are defined by eq. (58) and (59) respectively.

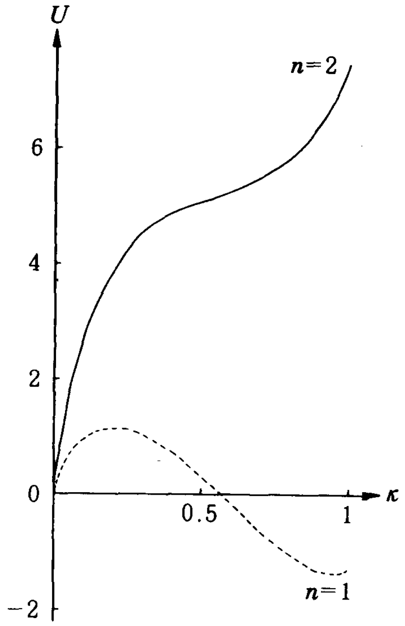


Fig. 11. The potential $U(\kappa)$ of sech(r^2)-type solution for $\alpha^2=2$ in the stationary $D=3$ case.
{ The dashed curve for $n=1$,
The solid curve for $n=2$.

We assume that the field has a spherical symmetry.

The corresponding field equation is reduced to

$$\Big[\frac{\partial^2}{\partial t^2}-\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)\Big]\phi(r,t)=-\alpha^{-2}\frac{dU}{d\kappa}\phi(r,t)$$
$$=-n\alpha^2\Big[n+2\{n-(n+1)\kappa^{1/n}\}\ \ln\frac{1+\sqrt{1-\kappa^{1/n}}}{\kappa^{1/2n}}-3\sqrt{1-\kappa^{1/n}}\Big]\phi(r,t)\ , \tag{70}$$

where

$$\kappa=\alpha^{-2}\phi^*(r,t)\,\phi(r,t). \tag{71}$$

The solution of eq. (70) is given by

$$\phi(r,t)=\alpha\exp(-i\alpha nt)\ \mathrm{sech}^n(\alpha^2r^2/2). \tag{72}$$

The field energy is

$$E=\tilde{T}_1+\tilde{T}_2+\tilde{V}_1$$
$$=4\pi\int_0^\infty dr\{\tilde{t}_2(r)+\tilde{t}_2(r)+\tilde{u}_1(r)\}, \tag{73}$$

where $\tilde{t}_1(r)$, $\tilde{t}_2(r)$ and $\tilde{u}_1(r)$ are defined by

$$\tilde{t}_1(r)=r^2\frac{\partial\phi}{\partial r}\frac{\partial\phi^*}{\partial r}$$
$$=n^2\alpha^6r^4\ \tanh^2(\alpha^2r^2/2)\ \mathrm{sech}^{2n}(\alpha^2r^2/2), \tag{74}$$

$$\tilde{t}_2(r) = r^2 \frac{\partial \phi}{\partial r} \frac{\partial \phi^*}{\partial r}$$

$$= n^2 \alpha^4 r^2 \ \mathrm{sech}^{2n}(\alpha^2 r^2/2) \tag{75}$$

and

$$\tilde{u}_1(r) = r^2 U(\kappa) \ . \tag{76}$$

Here, we introduce the following function;

$$\tilde{\omega}(r) = \tilde{t}_1(r) + 3\{\tilde{u}_1(r) - \tilde{t}_2(r)\} \ . \tag{77}$$

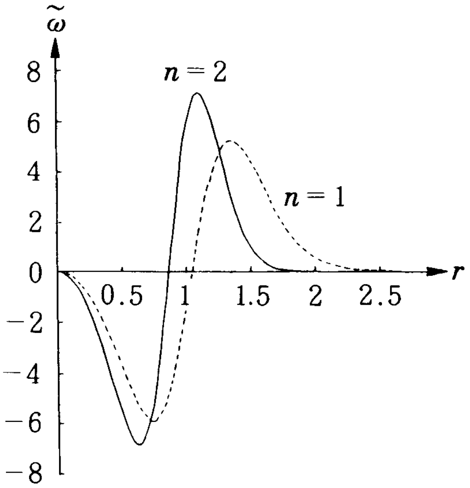


Fig. 12. $\tilde{\omega}(r) = \tilde{t}_1(r) + 3\{\tilde{u}_1(r) - \tilde{t}_2(r)\}$

{ The dashed curve for $n=1$ and $\alpha^2=2$,
The solid curve for $n=2$ and $\alpha^2=2$.

The shapes of $\tilde{\omega}(r)$ for $n=1$ and 2 are drawn in Fig. 12. We find that the integration of (77) is zero. Therefore, the virial relation of (30) for $D=3$ is satisfied;

$$\tilde{T}_1 + 3(\tilde{V}_1 - \tilde{T}_2) = 0 \ . \tag{78}$$

From (78), (73) is reduced to

$$E = 8\pi n^2 \alpha^4 \int_0^\infty dr \, \varepsilon(r) \ , \tag{79}$$

where

$$\varepsilon(r) = r^2 \ \mathrm{sech}^{2n} \ (\alpha^2 r^2/2) \ \{1 + \alpha^2 r^2 \tanh^2(\alpha^2 r^2/2)/3\} \ . \tag{80}$$

The field energy for $n=1$ and 2 is evaluated as

{ $E \simeq 54.1$ for $n=1$ and $\alpha^2=2$,
$E = 96.0$ for $n=2$ and $\alpha^2=2$.

5. DISCUSSION

The droplet solution[3] due to J.Werle have the following form;

$$\phi(r)=\begin{cases} A\ \exp(-i\mu t)\ (1-\alpha^2r^2)^{n/2} & \text{for } 0\leq\alpha r\leq 1 \\ 0 & \text{for } \alpha r>1, \end{cases} \tag{81}$$

where A, α and μ are certain constants with dimension $[L^{-1}]$,

$$\alpha^2=4\mu^2(n-2)n^{-1}(n+1)^{-2}$$

and $n\geq 4$. The corresponding potential is;

$$U(\kappa)=A^2\{\kappa-\frac{4(1-2/n)}{1-1/n^2}\kappa^{1-1/n}+\frac{4(1-2/n)}{(1+1/n)^2}\kappa^{1-2/n}\}, \tag{82}$$

where

$$\kappa=A^{-2}\phi^*\phi=(1-\alpha^2r^2)^n. \tag{83}$$

From (81), we see that the field is confined in the sphere (radius $1/\alpha$).

The algebraic solution[6] due to M.Umezawa is;

$$\phi(r)=1/(r^2+\lambda^2)^n, \tag{84}$$

where $n\geq 1$ and λ is a certain constant.

The potential is

$$U(\phi)=\frac{2n^2(2n-1)}{2n+1}\phi^{(2n+1)/n}-2n^2\lambda^2\phi^{(2n+2)/n}. \tag{85}$$

From (84), we see that the field is continuously decreasing from some finite value at $r=0$ to zero at $r=\infty$.

In our solutions (63) and (72), the fields are continuously decreasing from $r=0$ to $r=\infty$. They have a sharp boundary of the form $\exp(-n\alpha^2r^2/2)$, such that all multipole moments are finite. To represent baryons, our solutions would be much more preferable than the droplet-type or the algebraic solutions. Our potentials (56) and (69), however, are so much complex that we can not get easily the energy value by analytical integration, but we estimate them by numerical integrations.

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