Quaternionic Formulation of Classical Electrodynamics and Theory of Functions of a Biquaternion Variable.

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Abstract. Quaternionic formulation of classical electrodynamics by using "biq" (real part of a complex-quaternions) has been presented. Also, the solutions of Maxwell's equations have been given using regular functions of a biq variable.

Introduction : Complex quaternions are a powerful natural tool to describe mathematically classical electrodynamics and to obtain solutions of Maxwell's equations in vacuum by means of regular functions of a biq variable. [1, 2, 3]

§ 1. Biquaternions, biqs, Maxwell's equations and regular conditions for a function of a biq variable.

A complex quaternion (a biquaternion) Z is defined as:

(1. 1) $$Z=\sum_{\mu=0}^{3} e_\mu(x^\mu+iy^\mu)=X+iY, \quad (\mu=0, 1, 2, 3)$$
$$=x_0+\underline{x}+i(y_0+\underline{y})$$

where e_μ satisfy the following equations:

(1. 2) $$e_1e_2=-e_2e_1=ie_3, \text{ etc.}, \quad i=\sqrt{-1},$$
$$e_0=1, \quad e_k^2=1, \quad (k=1, 2, 3)$$

and

$$\underline{x}\equiv e_1x^1+e_2x^2+e_3x^3, \quad \underline{y}=e_2x^1+e_2x^2+e_3x^3,$$

is the "vector part" of $X=e_\mu x^\mu$ and we call x_0 the "scalar part" of X. We designate X and Y the "real part" and the "imaginary part" respectively, of Z.

"*Biqs*" :

When the imaginary part of a biquaternion is zero,

(1. 3) $$Z=X=\sum_{\mu=0}^{3} e_\mu x^\mu, \quad (x_\mu\in R)$$

we call specially a "biq" distinguishing it from a biquaternion.

Conjugation :

We define (1) hyper-conjugate, (2) complex-conjugate and (3) double-conjugate to $Z=x_0+\underline{x}+i(y_0+\underline{y})$ when

(1) $Z^+=x_0-\underline{x}+i(y_0-\underline{y})$,

(2) $Z^*=x_0+\underline{x}-i(y_0+\underline{y})$,

(3) $\bar{Z}=(Z^*)^+=(Z^+)^*$,

respectively.

For a biq X, we have $X^*=X$.

The conjugations of a product of two biquaternions A and B, we can show that

(1. 4) $(AB)^+=B^+A^+$,

(1. 5) $(AB)^*=B^*A^*$,

(1. 6) $\overline{(AB)}=\bar{A}\bar{B}$.

Norm.

The product of a biquaternion Z and its hyper-conjugate Z^+ define the norm of Z as

(1. 7) $$N(Z)=ZZ^+=Z^+Z$$
$$=(x_0+iy_0)^2-\sum_{k=1}^{3}(x_k+iy_k)^2$$

When $N(Z)=0$, we call Z a *zero divisor.*

Maxwell's equations in vacuum can be cast into the biquaternion form:

(1. 8) $DF^*(X)=4\pi I$,

where

(1. 9) $$F(X)=E+iH$$
$$=\sum_{\mu=0}^{3}e_\mu(E_\mu+iH_\mu)=\underline{E}+i\underline{H}.$$

(1. 10) $E_0=H_0=0$,

$$D\equiv\partial_0-\sum_{k=1}^{3}e_k\partial_k,\ \partial_\mu\equiv\frac{\partial}{\partial x_\mu},$$

(1. 11) $I=e_\mu i^\mu=i_0+\underline{i}$.

Equations (1. 8) through to (1. 11) is equivalent to

$$\operatorname{div}\underline{E}=-4\pi i_0,\quad \operatorname{div}\underline{H}=0,$$
$$\partial_0\underline{E}-\operatorname{curl}\underline{H}=-4\pi\underline{i},\quad \partial_0\underline{H}+\operatorname{curl}\underline{E}=0.$$

Regularity conditions.

Extending the regularity condition for functions of a quaternion variable [1] to that of a biq variable, we find the following conditions.

Let a function $\Phi(X)$ be a function of a biq varible $X=e_\mu x^\mu(\mu=0,1,2,3)$, $x^\mu\in R$, and

(1. 12) $\Phi(X)=u_0+i\underline{u}+i(v_0+\underline{v})$

where u_μ, v_μ are scalar, real functions of x^μ, and twice differentiable by x^μ. $\Phi(X)$ is called D left regular if $\Phi(X)$ satisfies

(1. 13) $D\Phi(X)=0$,

where D is defined by (1. 10). Writing in components:

$$(1.14)\quad \begin{aligned} &\partial_0 u_0-\operatorname{div}\underline{u}=0,\\ &\partial_0 v_0-\operatorname{div}\underline{v}=0,\\ &\partial_0\underline{u}-\operatorname{grad}u_0+\operatorname{curl}\underline{v}=0,\\ &\partial_0\underline{v}-\operatorname{grad}v_0-\operatorname{curl}\underline{u}=0. \end{aligned}$$

Comparing (1. 8) and (1. 13) if $I=0$, and identifying $\Phi(X)=F^*(X)$ together with imposing the conditions (1. 10), $\Phi(X)$ will give an electromagnetic field $F(X)$.

Thus, an electromagnetic field quantity and regular functions of a biq variable are intimately related. In the following we investigate the point.

Derivation of the regularity conditions.

To give a mathematical basis for the regularity conditions, we derive the regularity conditions from the condition that a functional is stationary with respect to the variation of the defining variable surface.

Let us take a four-dimensional volume of equation (1. 13):

(1. 15) $\int_{V^4} D\Phi(X)dv^4=\int_{S^3} dX^+\Phi(X)=0$,

where $dX^+\equiv e_\mu ds^\mu=e_\mu\lambda^\mu ds$, λ_μ: direction cosine of the normal to the surface element ds of the S^3. In (1. 15), we have used Gauss's integral formula. (1. 15) means that the integral of $\Phi(X)$ over S^3 is zero if no singularity ($D\Phi\neq 0$) is included inside and on the S^3.

§ 2. Solutions of Maxwell's equations by means of regular functions of a biq variable.

A straightforward method is to identify $F^*(X)$ as a D-left regular function by using (1. 8) and (1. 13) and to impose the "vector condition" (1. 10) on $\Phi(X)$ as an initial condition. Then, if $\Phi(X)$ satisfies at $x_0=0$: $u_0=v_0=0$, then $\Phi(X)$ satisfies the conditions for all time x_0.

We now introduce another method.

Let $F^*(X)$ be given by:

(2. 2) $\Phi(X)=(1/2)\ [\Phi(X)D]$,

where $\Phi(X)$ is a D left regular function and satisfies eq. (1. 13). Then, $F^*(X)$

satisfies Maxwell's equations and (1. 10) as shown below.

(2. 2) $DF^*(X)=(1/2)D[\Phi(X)D]=(1/2)[D\Phi(X)]D=0.$

From eq. (2. 1), the scalar part of (2. 1) is: $E_0=\partial_0 u_0-\mathrm{div}\, u=0$, $H_0=\partial_0 v_0-\mathrm{div}\, v=0$, by virue of (1. 14).

Thus, $F^*(X)$ *given by* (2. 1) *satisfies Maxwell's equations with* $I=0$.

Taking the complex conjugate of (2. 1) and using $D^*=D$, we get the following equation:

(2. 3) $\Phi^*(X)D=0.$

Adding $\Phi^*(X)D=0$ to the *rhs* of eq. (2. 1), we have for $F^*(X)$ the following:

(2. 4) $F^*(X)=(1/2)[\Phi(X)+\Phi^*(X)]D=U(X)D$

where $U(X)$ is the real part of $\Phi(X)=U(X)+iV(X)$.

Writing (2. 4) in component functions of $F(X)$, we have, by putting $U(X)=u_0+\underline{u}$, the tollowing equations:

$$(2.5)\quad \begin{aligned} E_0&=\partial_0 u_0-\mathrm{div}\,\underline{u}\,,\\ H_0&=0\,,\\ \underline{E}&=-\partial_0\underline{u}+\mathrm{grad}\, u_0,\\ \underline{H}&=\mathrm{curl}\,\underline{u}\,. \end{aligned}$$

Using the first equation of (1. 14), we find that u_0, $\underline{u}$ satisfy

(2. 6) $\partial_0 u_0-\mathrm{div}\,\underline{u}=0.$

Looking (2. 5) and (2. 6), we see that u_0, $\underline{u}$ play the role of the scalar and vector potentials of an electromagnetic field.

Thus, *we conclude that the real part of a D left regular function is the scalar + vector potential of an electromagnetic field.*

Several other methods to obtain the solutions of Maxwell's equations are available. [2]

(1) *Generation of regular functions.*

Example. Let $G(X)=u(x_0, x)+e\, v(x_0, x)$ be a regular function of a hyperbolic variable $X=x_0+ex$, $(e^2=1,\ e\neq 1)$, satisfies:

$$(2.7)\quad \begin{aligned} &\partial_0 u-\partial v=0,\ (\partial=\partial/\partial x,\ \partial_0=\partial/\partial x_0,)\\ &\partial_0 v-\partial u=0. \end{aligned}$$

Replacing e and x by the following:

(2. 8) $e=(e_1x_1+e_2x_2+e_3x_3)/x,\quad x=(x_1{}^2+x_2{}^2+x_3{}^2)^{\frac{1}{2}}$

we have a biq variable X from a hyperbolic variable X.

Then,

(2. 9) $\Phi(X)=\Box G(X),\quad \Box=DD^+=D^+D,$

satisfies D left as well as D right regulatity conditions.

Some of the functions $G(X)$ are X^n, $\exp(X)$ and many other functions which are analytic functions of a complex variable Z if the complex variable Z is replaced by the hyperbolic variable X.

Fueter's polynomial functions.

Generating functions are the following:

(2.10) $K^n(X, \underline{t}) \equiv (\underline{t}x_0 + (\underline{t} \cdot \underline{x}))^n$

$$= [t_1(x_0e_1 + x_1) + t_2(x_0e_2 + x_2) + t_3(x_0e_3 + x_3)]^n$$

Expanding K^n in power series of t_1, t_2, t_3, we define P-functions as

(2.11) $K^n(X, t) = \sum_{n_1n_2n_3}^{\Sigma n_i = n} n!\, P_{n_1n_2n_3}(X) t_1^{n_1} t_2^{n_2} t_3^{n_3},$

We can show easily that $DK^n(X) = K^n(X, \underline{t})D = 0$, and since t_1, t_2, t_3 are independent parameters, each terms $P_{n_1n_2n_3}(X)$ are both side D regular:

(2.12) $DP_{n_1n_2n_3}(X) = P_{n_1n_2n_3}(X)\, D = 0.$

Exponential functions.

(2.13) $\exp[iK(X, \underline{t})] = \sum_{n=0}^{\infty} (n!)^{-1}(i)^n K^n(X, \underline{t})$

Then,

$$D \exp[iK(X, \underline{t})] = \exp[iK(X, \underline{t})]\; D = 0.$$

Since $\exp[iK(X, \underline{t})]$ is a special solution of $D\, \Phi(X) = 0$, a general solution is given by

(2.14) $\Phi(X) = \int\int\int_{-\infty}^{+\infty} \exp[iK(X, \underline{t})]\; A(t_1, t_2, t_3)\; dt_1dt_2dt_3$

Putting the initial condition:

(2.15) $\Phi(X) = G(\underline{x})$,

we have

(2.16) $G(\underline{x}) = \int\int\int_{-\infty}^{+\infty} \exp[i(x_1t_1 + x_2t_2 + x_3t_3)]\; A(t_1, t_2, t_3)\, dt_1dt_2dt_3,$

Inverting the integral by Fourier integral:

(2.17) $A(t_1\, t_2\, t_3) = (2\pi)^{-3} \int\int\int_{-\infty}^{+\infty} \exp[-i(\underline{t} \cdot \underline{\tau})[G(\underline{x}, \underline{t}) d\tau_1 d\tau_2 d\tau_3$

we have:

(2.18) $\Phi(X) = \frac{1}{(2\pi)^3} \int\int\int_{-\infty}^{+\infty} dt^3\, d\tau^3\; \exp[\underline{t}x_0 + i(\underline{x} - \underline{\tau}) \cdot \underline{t}]\; G(\underline{\tau}).$

Polynomial series expansion.

We can easily show the following relations;

$$\begin{aligned} \partial_1 P_{n_1n_2n_3}(X) &= P_{n_1-1n_2n_3}(X), \\ (2.19) \quad \partial_2 P_{n_1n_2n_3}(X) &= P_{n_1n_2-1n_3}(X), \\ \partial_3 P_{n_1n_2n_3}(X) &= P_{n_1n_2n_3-1}(X), \end{aligned}$$

Thus, we have the following;

$$(2.20) \quad \partial_1^{r_1}\partial_2^{r_2}\partial_3^{r_3} P_{n_1n_2n_3}(X) = \delta_{r_1n_1}\delta_{r_2n_2}\delta_{r_3n_3} \qquad (r_1+r_2+r_3=n_1+n_2+n_3).$$

A D left regular function $G(X)$ can be expanded as:

$$(2.21) \quad G(X) = \sum_{n=0}^{\infty} \sum_{\Sigma n_i=n}^{n_1n_2n_3} P_{n_1n_2n_3}(X)\, C_{n_1n_2n_3},$$

where

$$(2.22) \quad C_{n_1n_2n_3} = \partial_1^{n_1}\partial_2^{n_2}\partial_3^{n_3} G(X)|_{X=0}.$$

Putting $x_0=0$ and using the relations: $P_{n_1n_2n_3}(X)_{\,x_0=0} = x_1^{n_1}x_2^{n_2}x_3^{n_3}$, we find

$$(2.23) \quad G(X)|_{x_0=0} = G(\underline{x}) = \sum_{n=0}^{+\infty} \sum_{\Sigma n_i=n}^{n_1n_2n_3} x_1^{n_1}x_2^{n_2}x_3^{n_3}\, C_{n_1n_2n_3}$$

By Taylor expansion of $G(X)$, we have

$$(2.24) \quad C_{n_1n_2n_3} = \partial_1^{n_1}\,\partial_2^{n_2}\,\partial_3^{n_3}\, G(\underline{x})|_{\underline{x}=0}.$$

Theorem of residues.

Theorem of residues corresponding to that in the complex variable theory is somewhat different from that in the complex variable theory because of the existence of zero divisors in the biq variable theory. We may describe the theorem briefly as follows [2].

Let $F(X)$ be D^+ right regular and S^3 be a closed hypersurface lying entirely in the regular domain V^4 and contains a point A inside, then

$$(2.25) \quad F(A) = (8\pi)^{-2}\int_{S_F^3} F(X)dZ^+(\Box(X-A)^{-1})$$

where the integration by dZ^+ is taken over S_F^3 which is defined by

$$(2.26) \quad S_F^3 = [\,[(x_1^2+x_2^2+x_3)^{1/2}-x_0]^2+y_0^2=\varepsilon^2,\ x_0=\text{const},\ \ \varepsilon=0].$$

We may derive the expressions for the retarded potentials and Lienard-Wiehert's potential for a point charge. [2]

§ 3. Special theory of relativity.

We now describe briefly the special theory of relativity in the biquaternion formulation.

1. *Velocity*: Let a biq $X=x_0+\underline{x}$, $x_0\equiv ct$ be the coordinate biq of a particle. The four velocity U is defined as

$$(3.1) \quad U = dX/cd\tau, \quad (c\text{: light velocity})$$

where τ is a scalar parameter chosen such that U is a unit biq:

(3.2) $N(U)=UU^{+}=1.$

τ is called the proper time of the particle. From (3.1) and (3.2) we have

(3.3) $U=(dx_0/cd\tau)\,[1+(d\underline{x}/dx_0)]=\gamma(1+\underline{u}/c)$

where $\gamma=(1-u^2/c^2)^{-1/2},\ x_0=c\gamma\tau,\ \underline{u}=c(d\underline{x}/dx_0).$

From (3.2), we have

(3.5) $$c^2d\tau^2=N(U)c^2d\tau^2=(dX)(dX)^{+}$$
$$=dx_0{}^2-dx_1{}^2-dx_2{}^2-dx_3{}^2.$$

2. *Lorentz transformations.*

A general linear homogeneous transformation which maps biqs into themselves and preserves the norm of X can be expressed as:

(3.6) $X\to X'=A^*XA,\ N(A)=1,$

where A is a biquaternion

(3.7) $A=c_0+\underline{c}+i(d_0+\underline{d}).$

The transformation can be expressed by a product transformation of a pure Lorentz transformation:

(3.8) $X_1=R^*(\pm)XR(\pm),\ R^*(\pm)=R(\pm),\ N(R(\pm))=1.$

and a space rotation:

(3.9) $X_2=Q^*XQ,\ \bar{Q}=Q,\ N(Q)=1.$

We can show that a unit biquaternion A can be expressed as a product of a unit biq $R(\pm)$ and a unit quaternion Q as follows:

(3.10) $A=R(+)Q=QR(-),$

where

(3.11) $A=c_0+\underline{c}+i(d_0+\underline{d}),$

(3.12) $$Q=(c_0+i\underline{d}),\ q=(c_0{}^2+d^2)^{-1/2},$$
$$R(\pm)=q+(c_0\underline{c}+d_0\underline{d}\pm[\underline{c}\times\underline{d}]q)$$

where

$$[\underline{c}\times\underline{d}]=(c_2d_3-c_3d_2)e_1+(c_3d_1-c_1d_3)e_2+(c_1d_2-c_2d_1)e_3$$

Thus, (3.6) is a product transformation:

(3.13) $X'=(R(+)Q)^*X(R(+)Q)=Q^*(R^*(+)XR(+))Q,$

and

$$X'=(QR(-))^*X(QR(-))=R^*(-)(Q^*XQ)R(-).$$

Thus, (3.13) and (3.14) are a successive transformation of a space rotation around the axis $\underline{d}$ by an angle $\omega=2\tan^{-1}(d/c_0)$, followed (anteceded) by a pure Lorentz transformation which is moving with a relative velocity $\underline{v}_-(\underline{v}_+)$ relative to the system $S(X)$ given by

$$V_+=\gamma_+(1-\underline{v}_+/c)=AA^*=R^*(+)R(+),$$
(3.15) $$V_-=\gamma_-(1-\underline{v}_-/c)=A^*A=R^*(-)R(-),$$
$$\gamma_\pm=(1-v_\pm^2)^{-1/2},\ \underline{v}_\pm=(V_\pm-V_\pm^+)(V_\pm+V_\pm^+)^{-1}$$

which is moving with a velocity $\underline{v}_-(\underline{v}_+)$ relative to $S(X)$.

3. *Energy-momentum, acceleration and Lorentz force.*

The energy-momentum biq P of a particle is defined as:

(3.17) $P=mc^2U=E+\underline{p}c.$

The acceleration biq is given by:

(3.18) $dU/cd\tau=c^{-2}(d^2X/d\tau^2)=\gamma(dU/dt)c^{-1}=(\gamma/c)d[\gamma(1-\underline{u}/c)]/dt.$

The Lorentz force biq K is defined as:

(3.19) $(e_0/2)(FU+UF^*)=e_0\gamma[(\underline{E}\cdot\underline{u})/c+[\underline{u}\times\underline{H}]/c]=K$

where $F=\underline{E}+i\underline{H}$ is the electromagnetic field biquaternion.

Since K is a force and can be defined by the acceleration as:

(3.20) $K=dP/d\tau=mc^2dU/d\tau.$

Since $N(U)=1$, $(dU/d\tau)U^++U(dU^+/d\tau)=0$, we have $KU^++UK^+=0$. Putting $K=k_0+\underline{k}$, we have

(3.21) $k_0=(\underline{k}\cdot\underline{u})/c^2.$

The equation of motion of a charged particle is, by (3.19) and (3.20), given by:

(3.22) $K=m_0(d^2X/d\tau^2)=(e_0/2)(F(dX/d\tau)+(dX/d\tau)F^*).$

The solutions $dX/d\tau$ and X of (3.22) can be obtained for special cases of F as described below.

Example 1.

F is uniform and constant.

The solution of (3.22) is

(3.23) $cU=dX/d\tau=A^*(\tau)XA(\tau),$

where

$$A(\tau)=\exp[(e_0/2m_0)\int^\tau F(\tau)d\tau]$$

(3.24) $U_0=(dX/d\tau)_{\tau=0}=\gamma_0(1+\underline{u}_0/c)$

(3.25) $X=\int_0^\tau A^*(\tau')U_0A(\tau')d\tau'+C.$

For an arbitrary biquaternion $Z=x_0+\underline{x}+i(y_0+\underline{y})$

(3.26) $\exp(Z)=\exp(x_0+iy_0)\ [\cosh(\alpha+i\beta)+\underline{e}\sinh(\alpha+i\beta)]$

where

α, β are real numbers such that

(3.27) $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=2^{-1/2}[((x^2-y^2)^2+4(\underline{x}\cdot\underline{y})^2)^{1/2}\pm(x^2-y^2)]^{1/2}$,

Example 2.

In case $F(X)$ is given by an explicit function of X.

By solving (3.22), we have

(3.29) $m_0(dX/d\tau) = (e_0/2)\,[\int^X F(X)dX+\int^X dXF^*(X)]+\text{const.}$

and

(3.30) $[\frac{e_0}{2m_0}](\tau-\tau_0)=\int_{X_0}^X [\int^X F(X')dX'+\int^X dX'\,F^*(X')+\text{const}]^{-1}dX$

Integrating the r. h. s. of (3.30), we obtain τ as a function of X: $\tau=G(X, c_1, c_2)$. By solving the equation, we obtain X as a function ot τ, (c_1, c_2 are integration constants).

Energy-momentum tensors.

From equations (3.22) and (1.11): $I=e_0U$, we have

(3.31) $K=(1/2)(FI+IF^*)$

$=(1/8\pi)(F(DF^*)+(DF^*)^*F^*)$

$=(1/8\pi)(FDF^*)=(1/8\pi)\partial^\mu(Fe_\mu F^*)$,

where we have used eq. (1.18) and $I^*=I=(DF^*)^*$.

(3.3) can be cast into the following form:

(3.31) $K=\partial^\mu T_\mu$,

$T_\mu=(1/2)(Fe_\mu F^*)=e_\lambda t^\lambda{}_\mu$.

then, $t^\lambda{}_\mu$ is the energy-momentum tensor of an electromagnetic field.

Appendix. An extension of the conformal mappings in the complex variable theory to a biq variable theory.

By the process described in Example in §2 and the potential function method, we can map a D left regular function and a potential function trom a biq variable to another, preserving the regularity cnoditions as well as the vector condition. Using the mapping of the hyperbolic variables X by an analytic function of a hyperbolic variable $f(Y)$:

(A.1) $X=f(Y)=X(Y)$,

where

(A.2) $X=x_0+e_x x,\quad Y=y_0+e_y y$.

We consider the function $f(Y)=X$ as a biq mapping from X to Y when x, e_x, y, and e_y are replaced by the following relations:

$$e_x=(e_1x_1+e_2x_2+e_3x_3)/x,\quad x=(x_1^2+x_2^2+x_3^2)^{1/2},$$

and similar relations for e_y and y.

Let $G(X)$ be a regular function of a hyperbolic variable considered in Example in § 2, then by (A. 1),

$$G(Y)=G(F(Y))$$

can be considered as a D both side regular function of Y.

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