Network Representasion of Quasi-exponentially Tapered Line

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(Received September 27, 1984)

Abstract—To simplify calculations for pseudo-distortionless transmission lines, a new circuitrical approach is presented. On the process of solving the line equations the Liouville transformation plays an important role, in the sense that the Liouville normal form for the original line equation yields explicit solution for a variety of taper of transmission lines. A transmission line section can be presented at all frequencies by the lumped networks, provided the rational fraction expansion of the immittance functions can be obtained. This paper presents complete network functions for quasi-exponentially tapered lines.

Apparent Lossless Line

Consider a tapered pseudo-distortionless transmission line of length ℓ, whose distributed inductance, resistance, capacitance and conductance per unit length at position x are designated by $l(x)$, $rl(x)$, $c(x)$ and $gc(x)$, respectively, and are at least twice continuously differentiable. A pair of line equations relating the Laplace-transformed voltage $v(x, s)$ and the current $i(x, s)$ along the pseudo-distortionless line is

$$\begin{aligned} v'_x(x, s) &= -(s+r)\ l(x)\ i(x, s) \\ i'_x(x, s) &= -(s+g)\ c(x)\ v(x, s), \end{aligned} \tag{1}$$

where s is the complex frequency variable ($s=\sigma+j\omega$). In (1) and throughout this paper, the prime is used to denote differentiation so that y'_x denotes ∂_y/∂_x, etc.

Making a group of transformations,

$$X=\int_0^x \sqrt{lc}\, dx \Big/ \int_0^\ell \sqrt{lc}\, dx \tag{2}$$

$$S=\sqrt{(s+r)(s+g)}\int_0^\ell \sqrt{lc}\, dx \tag{3}$$

$$V=\sqrt{s+g}\, v \tag{4}$$

$$I = \sqrt{s+r}\, i \tag{5}$$

$$L = 1/C = \sqrt{l/c}\,. \tag{6}$$

reduces (1) to a pair of line equations,

$$\begin{aligned} V'_x(X,S) &= -S\,L(X)\;I(X,S) \\ I'_x(X,S) &= -S\,C(X)\;V(X,S). \end{aligned} \tag{7}$$

A series of transformations in (2) - (6) is regarded as a kind of Liouville transformation. By applying the Liouville transformation to an 'original' pseudo-distortionless transmission line of length ℓ, we obtain an 'apparent' lossless line of length 1. In order to analize a transmission line simply, the Liouville transformation becomes a very useful method. Combining the equations in (7) leads to the following differential equations in normal form,

$$(\sqrt{C(X)}\;V(X,S))''_{xx} - (S^2 + \triangle(X)^2)\sqrt{C(X)}\;V(X,S) = 0 \tag{8a}$$

where $\triangle^2(X) = (\sqrt{C(X)})''_{xx} / \sqrt{C(X)}$

and

$$(\sqrt{L(X)}\;I(X,S))''_{xx} - (S^2 + \triangle(X)^2)\sqrt{L(X)}\;I(X,S) = 0 \tag{8b}$$

where $\triangle^2(X) = (\sqrt{L(X)})''_{xx} / \sqrt{L(X)}$.

Since $L(X)$ and $C(X)$ are positive, $\triangle^2(X)$ is real. The expressions for $\triangle^2(X)$ are changed into differential equations in normal form,

$$(\sqrt{C(X)})''_{xx} - \triangle^2(X)\sqrt{C(X)} = 0 \qquad \text{(for (8a))} \tag{9a}$$

and

$$(\sqrt{L(X)})''_{xx} - \triangle^2(X)\sqrt{L(X)} = 0 \qquad \text{(for (8b))} \tag{9b}$$

Every $\triangle(X)$ for which both (8) and (9) can be solved leads to an explicit solution for $V(X,S)$ and $I(X,S)$. A generalized expression of solution for the equations in (8) can be written as

$$\begin{vmatrix} V(X,S) \\ I(X,S) \end{vmatrix} = Q(X,S)\,K$$

where Q denotes a two-by-two matrix, and K is a two-entry constant column matrix. Since K can be determined uniquely by terminal conditions, Q must be nonsingular. The voltage and current at near end thus can be represented in terms of the volage and current at far end:

$$\begin{vmatrix} V(0,S) \\ I(0,S) \end{vmatrix} = Q(0,S)\;Q^{-1}(1,S) \begin{vmatrix} V(1,S) \\ I(1,S) \end{vmatrix} \tag{10}$$

A cascade matrix thus can be expressed as

$$\begin{vmatrix} A(S) & B(S) \\ C(S) & D(S) \end{vmatrix} = Q(0,S)\;Q^{-1}(1,S) \tag{11}$$

Quasi-exponentially Tapered Lines

Let $\alpha(X)$ denote impedance ratio to near end:

$$\alpha(X)=L(X)/L(0) \tag{12}$$

A quasi-exponeotially tapered line can be defined to be the transmission line for which either $\alpha(X)$ or $1/\alpha(X)$ is equal to function,

$$f(X)=\mathrm{csch}^2\,\delta(\sqrt{f(1)}\sinh\delta X+\sinh\delta(1-X))^2 \tag{13}$$

The term δ is a parameter which characterizes the curve, $f(X)$. For various real values of δ^2, $f(X)$ represents a group of curves which are drawn through the two fixed points, $f(0)$ and $f(1)$. The group of curves shows to include the uniform, exponential, quadratic, hyperbolic and trigonometric curves. In the case of $\delta<0$, taper is squared-trigonometric. In the case of $\delta=0$, taper is quadratic. In the case

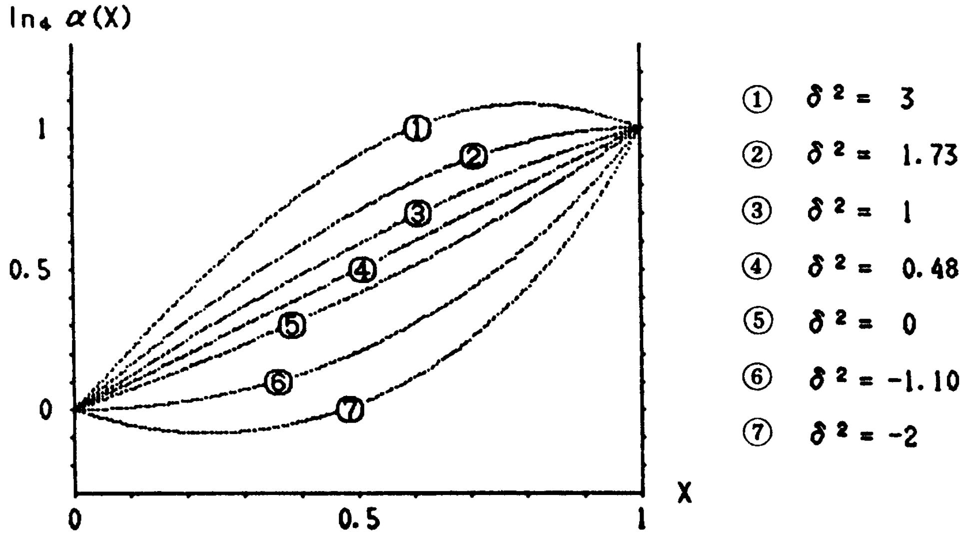


Fig. 1 Examples of quasi-exponential taper of $\alpha(X)=1/f(X)$.

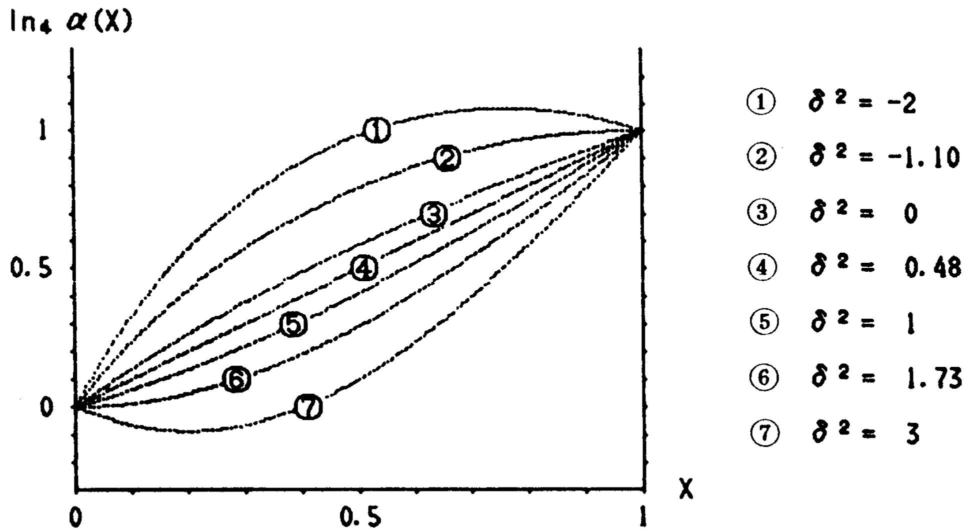


Fig. 2 Examples of quasi-exponential taper of $\alpha(X)=f(X)$.

of $\delta>0$, taper is squared-hyperbolic. Especially, taper of $\delta=\ln\alpha(1)/2$ is exponential, and taper of $\delta=\ln\alpha(1)/2=0$ is uniform. Various examples of quasi-exponential taper are shown in Fig. 1 and Fig. 2.

I) Taper of $\alpha(X)=1/f(X)$

In a transmission line section of $\alpha(X)=1/f(X)$, only parameter $B(S)$ is complicated. It thus is convenient to represent such line section in trems of open-circuited impedance functions. The impedance matrix $\mathbf{Z}(S)$ is

$$\mathbf{Z}(S)=\begin{vmatrix} Z_{11}(S) & Z_{12}(S) \\ Z_{21}(S) & Z_{22}(S) \end{vmatrix} \tag{14}$$

where
$$Z_{11}(S)= L(0)(\Gamma(S)\ \coth\Gamma(S)+(\ln\sqrt{L(0)})'_x)/S$$
$$Z_{22}(S)= L(1)(\Gamma(S)\ \coth\Gamma(S)-(\ln\sqrt{L(1)})'_x)/S$$
$$Z_{12}(S)= Z_{21}(S)=\sqrt{L(0)L(1)}\ \Gamma(S)\ \mathrm{csch}\Gamma(S)/S$$
$$\Gamma(S)=\sqrt{S^2+\delta^2}$$

By making use of the rational fraction expansions,

$$\coth \chi= 1/\chi+ \sum_{n=1}^{\infty} 2\chi/(\chi^2+(n\pi)^2) \tag{15}$$

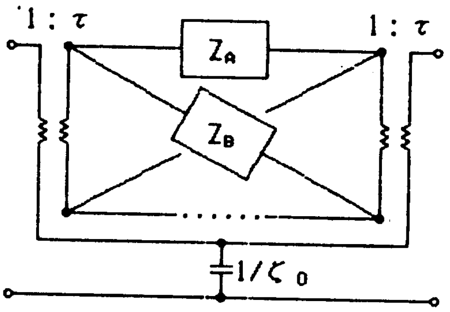
$$\mathrm{csch}\ \chi= 1/\chi+ \sum_{n=1}^{\infty} (-1)^n 2\chi/(\chi^2+(n\pi)^2) \tag{16}$$

the impedance matrix in (14) can be expanded as

$$\mathbf{Z}(S)=\mathbf{Z}_1(S)+\mathbf{Z}_2(S) \tag{17}$$

where
$$\mathbf{Z}_1(S)=\frac{1}{S/\zeta_0}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$
$$\mathbf{Z}_2(S)=\sum_{n=1}^{\infty}\frac{1}{S/\zeta_n+\dfrac{1}{S\zeta_n/\lambda_n}}\begin{vmatrix} 1/\tau^2 & (-1)^n \\ (-1)^n & \tau^2 \end{vmatrix}$$
$$\zeta_0= \sqrt{L(0)L(1)}\ \delta\ \mathrm{csch}\ \delta$$
$$\zeta_n=\sqrt{L(0)L(1)}\ 2(n\pi)^2/((n\pi)^2+\delta^2)$$
$$\lambda_n=(n\pi)^2+\delta^2$$
$$\tau = \sqrt[4]{\alpha(1)}.$$

This class of quasi-exponentially tapered line section thus can be syntesized as series connection of two-port $\mathbf{Z}_1(S)$ and two-port $\mathbf{Z}_2(S)$. The two-port $\mathbf{Z}_1(S)$ has only a shunt arm of capacitance $1/\zeta_0$. For $\mathbf{Z}_2(S)$, the term $Z_{21}(S)$ consists of alternate positive and negative signs. The two-port $\mathbf{Z}_2(S)$ thus can be synthesized as a symmetrical lattice cascaded with two ideal transformers with a turns ratio of $1:\tau$. Finally, a lumped equivalent two-port for the quasi-exponentially tapered line section can be shown as in Fig. 3.



$$Z_A=\sum_{\substack{n=1\\odd}}^{\infty}\frac{1}{S/\zeta_n+\dfrac{1}{S\zeta_n/\lambda_n}}$$

$$Z_B=\sum_{\substack{n=2\\even}}^{\infty}\frac{1}{S/\zeta_n+\dfrac{1}{S\zeta_n/\lambda_n}}$$

Fig. 3 Lumped equivarent circuit for a line of $\alpha(X)=1/f(X)$.

II) Taper of $\alpha(X)=f(X)$

In a transmission line section of $\alpha(X)=f(X)$, only the parameter $C(S)$ is complicated. It thus is convenient to represent such line section in terms of short-circuited admittance functions. The admittance matrix $\mathbf{Y}(S)$ is

$$\mathbf{Y}(S)=\begin{vmatrix} Y_{11}(S) & Y_{12}(S) \\ Y_{21}(S) & Y_{22}(S) \end{vmatrix} \tag{18}$$

where $Y_{11}(S)=C(0)\,(\Gamma(S)\coth\Gamma(S)+(\ln\sqrt{C(0)})'_x)/S$

$Y_{22}(S)=C(1)\,(\Gamma(S)\coth\Gamma(S)-(\ln\sqrt{C(1)})'_x)/S$

$Y_{12}(S)=Y_{21}(S)=\sqrt{C(0)C(1)}\;\Gamma(S)\,\mathrm{csch}\,\Gamma(S)/S$

Applying the rational fraction expansions in (15) - (16) to (18) yields

$$\mathbf{Y}(S)=\mathbf{Y}_1(S)+\mathbf{Y}_2(S) \tag{19}$$

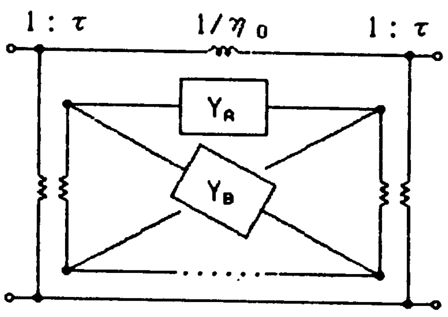
where $\mathbf{Y}_1(S)=\dfrac{1}{S/\eta_0}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$

$$\mathbf{Y}_2(S)=\sum_{n=1}^{\infty}\frac{1}{S/\eta_n+\dfrac{1}{S\eta_n/\lambda_n}}\begin{vmatrix} \tau^2 & -(-1)^n \\ -(-1)^n & 1/\tau^2 \end{vmatrix}$$

$\eta_0=\sqrt{C(0)C(1)}\;\delta\,\mathrm{csch}\,\delta$

$\eta_n=\sqrt{C(0)C(1)}\;2(n\pi)^2/((n\pi)^2+\delta^2)$

This class of quasi-exponentially tapered line section thus can be synthesized as shunt connection of two-port $\mathbf{Y}_1(S)$ and two-port $\mathbf{Y}_2(S)$. The two-port $\mathbf{Y}_1(S)$ has only a series arm of inductance $1/\eta_0$. For $\mathbf{Y}_2(S)$, the term $Y_{12}(S)$ consists of



$$Y_A=\sum_{\substack{n=2\\even}}^{\infty}\frac{1}{S/\eta_n+\dfrac{1}{S\eta_n/\lambda_n}}$$

$$Y_B=\sum_{\substack{n=1\\odd}}^{\infty}\frac{1}{S/\eta_n+\dfrac{1}{S\eta_n/\lambda_n}}$$

Fig. 4 Lumped equivarent circuit for a line of $\alpha(X)=f(X)$.

alternate positive and negative signs. The two-port $\mathbf{Y}_2(S)$ thus can be synthesized as a symmetrical lattice cascaded with ideal transformers with a turns ratio of $1:\eta$. Finally, a lumped equivalent two-port for the quasi-exponentially tapered line section can be shown as in Fig. 4.

Conclusion

It became clear that a group of transformations in (2) - (6) can reduce any tapered pseudo-distortionless transmission line to a normalized "'apparent' lossless line, and that a quasi-exponentially tapered line of finite length can be synthesized by either open-circuited impedance functions or short-circuited admittance functions at all frequencies. Likewise, arbitrarily tapered lines will be represented by lumped networks, provided rational fraction expantion of immittance functions can be obtained. We leave a detailed discussion about perfect proof for arbitorarily tapered lines.

Acknowledgment

The author wishes to thank Prof. J. Ishii, Kinki University, for many useful discussions and suggestions relating to the work.

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