On Regular Functions of a Nonalternative Hypercomplex Variable

K. IMAEDA* and Mari IMAEDA.**

* *Department of Electronic Science, Okayama University of Science, 1-1 Ridaicho, Okayama, 700, Japan.*
** *I.F.U.P., 31700 Cheswick Place, Solon, Ohio, 44139, U. S. A.*

(Received September 27, 1984)

Abstract

For the algebra of hypercomplex numbers in our theory, neither the associative law nor the alternative law is assumed. We have assumed the power associative law for the algebra and the anticommutation relations for the units i_k ($k=1, \ldots, 2s-1$): $i_l i_k + i_k i_l = -2\delta_{kl}$. There are several subvariables which do not contain certain coordinate variables x_k. Accordingly, there are several regularity conditions each of which is applicable to functions of a specific variable. It is shown that a function $G(X) \equiv \Box^{s-1} F(X)$ is a regular function of a hypercomplex variable $X = x_0 + i_1 x_1 + \ldots + i_{2^s-1} x_{2^s-1}$, $\Box \equiv \sum_{k=0}^{2^s-1} \frac{\partial^2}{\partial x_k^2}$ and assuming that $F(X)$ is a regular function of a complex variable $X = x_0 + ix$. We have derived the integral theorems, regular polynomial functions, exponential functions and Fourier integral theorems. The results may be of use even to those functions of a Clifford variable and to those of an octonion variable which are alternative or even associative.

I. Introduction.

In a previous paper,(1) we have developed a theory of functions of an octonion variable. To extend the theory to a higher hypercomplex variable which is not-associative and not-alternative, one would expect to confront with a great difficulty in constructing a theory of a nonalternative hypercomplex variable compared with an alternative hypercomplex variable such as an octonion variable as we have discussed before.

However, as has been illustrated in this paper, the theory of regular functions of a quaternion variable(2) or an octonion variable we have developed in previous papers can be extended without a serious difficulty in constructing a theory of a nonalternative hypercomplex variable including "16-nions" (not-alternative) and the higher hypercomplex variables such as "2^n—nions" for $n=5, 6$, etc., which are constructed from octonions by a successive application of the "Cayley-Dickson" process [3] for some kind of hypercomplex functions.

In this paper, we have developed a theory of regular functions of a hypercomplex variable which is not-alternative. However, the theory can be applied to hypercomplex variable not only those of a Clifford variable but also of a hypercomplex variable including an octonion variable.

II. Algebra of hypercomplex numbers "2^n-nions".

We define *a* 2^n-nions A by *a* pair of 2^{n-1}-nions a_1, a_2 and the sum and the product of two 2^n-nions A and B are as follows.

(1) $A=(a_1, a_2), \quad B=(b_1, b_2),$

$A\pm B=(a_1\pm b_1, a_2\pm b_2),$

$A(B+C)=AB+AC, \quad (B+C)A=BA+CA,$

and

(2) $AB=(a_1b_1-\bar{b}_2a_2, a_2\bar{b}_1+b_2a_1)$

where $\bar{b}_1$, $\bar{b}_2$ are the conjugate of b_1, b_2. The conjugate $\bar{A}$ of A is defined as:

(3) $\bar{A}=(\bar{a}_1, -a_2).$

In the following, "2^n-nions" for $n=0, 1, 2, 3 \ldots$ are described.

(I) "2^0-nions" are defined as real numbers: the conjugate $\bar{r}$ of r is the same as r: $\bar{r}=r$.

(II) "2^1-nions" are complex numbers:

A complex number c is defined as a pair of real numbers r_1, and r_2. Let

$c=(r_1, r_2), \; c'=(r_1', r_2'), \quad r_1, r_2, r_1', r_2' \in R,$

then

$c\pm c'=(r_1\pm r_1', r_2\pm r_2'),$

and by (1) and (3):

(4) $c\,c'=(r_1r_1'-\bar{r}_2'r_2, r_2\bar{r}_1'+r_2'r_1).$

The multiplication rule given by (4) is the same as that of complex numbers when we put $(r_1, r_2)=r_1+r_2i$, $i^2=\sqrt{-1}$.

(III) "2^2-nions": Quaternions.

A quaternion Q is defined as a pair of complex numbers c_1, c_2

$$Q=(c_1, c_2).$$

The multiplication of two quaternions Q, Q' do not satisfy commutation law:

$$Q\,Q'-Q'\,Q\neq 0.$$

Even though complex numbers are commutative, a complex number is not equal to its conjugate: $\bar{c}_i \neq c_i$, two quaternions in general, are not commutative:

$$Q\,Q' \neq Q'\,Q.$$

(IV) "2^3-nions": octonions.

An octonion A is defined as a pair of quaternions a_1, a_2 and the multiplication rule of two octonions derived by Cayley-Dickson process (4) results in that octonions are neither commutative nor associative.

Using the multiplication rule (4) and using that two quaternions are not commutative, we can show the following relations: Let A, B, C be octonions, the associative law does not hold:

(6) $(AB)C \neq A(BC)$.

(V) "16-nions".

Using the multiplication law (1) and that octonions do not obey the associative law (6), we can show, for 16-nions, the alternative law is violated:

(8) $A^2B \neq A(AB)$.

By equations (8) and (2), we can show that 16-nions are no longer a division algebra[(4)] and are not a normed algebra[(4)].

(9) $|AB| \neq |A|\,|B|, \quad |A| = A\bar{A}$.

We may proceed to define a 2^{n+1}-nion by a pair of 2^n-nions through the process (1) and (2), starting from real numbers by a repeated procedure (1).

For "2^n-nions", the basis units $i_0, i_1, \dots, i_{2^n-1}$ satisfy:

(10) $i_j i_k + i_k i_j = -2\delta_{jk} i_0\,,\ i_0 = 1\,;\ 1 \leqq j,\ k \leqq 2^n - 1$.

In the following, we confine ourselves to consider hypercomplex variable the units of which satisfy the relations (10) and the algebra is assumed to be neither associative nor alternative but power associative: $A^n A^m = A^{n+m}$.

III. Regularity conditions and integral theorems.

(A) Regularity conditions.

We define several regularity conditions for functions of a hypercomplex variable belonging to an algebra α.

We consider functions of a hypercomplex variable which takes the value of hypercomplex numbers belonging to an algebra α (we do not assume alternative law for the algebra: $A^2B = A(AB)$). The algebra has n units: $\mathrm{i}_0 = 1, \mathrm{i}_1, \mathrm{i}_2, \dots, \mathrm{i}_{n-1}$ satisfying:

(13) $i_j i_k + i_k i_j = -2\delta_{jk}$, for $1 \leqq j,\ k \leqq n-1$.

We consider functions of a single hypercomplex variable X_m expressed as:

(14) $X_m = x_0 + \sum_{k=1}^{m-1} i_k x_k,\ 2 \leqq m \leqq n,$

and the domain of $(x_0, x_1, \dots, x_m)$ is in m-dimensional subspace of the n-dimensional space $(x_0, x_1, \dots, x_m, \dots x_{n-1})$.

When $m=n$, the variable X_m is in a region of the n-dimensional space but for $m<n$, the region of space is confined in the region of the m-dimensional subspace of X_n.

As for example, in the case of octonions, the space of eight dimension, we may take the variable X_m for $m=2, 3, 4, \dots, 8$, which are defined in 2, 3, ..., 8 dimensional space, respectively.

In the following we consider functions of a single hypercomplex variable X_m where m is a fixed positive integar when we deal with a function. We do not assume neither associative law nor alternative law for the hypercomplex variable. We assume the "power associative law" for the variable X_m:

(15) $X_m{}^s X_m{}^r = X_m{}^{s+r}$,

for any integars r and s and the commutation relations (13).

Now we introduce a regularity condition as follows:

In the following we omit writing the suffices m of X_m and D_m but write simply X and D. We understand that when $m<n$, functions we are dealing are in the m dimensional subspace.

Let $F(X)$ be a function of a hypercomplex variable X, then the regularity condition is expressed as:

(16a) $DF(X)=0$, (left D regular),

(16b) $F(X)D=0$, (right D regular),

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{k=1}^{m-1} i_k \frac{\partial}{\partial x_k},$$

in the m dimensional domain d^m.

When $m=n$, (16a, b) are called "regularity condition in the n-dimensional space" or "full regularity condition" and when $m<n$, (16a, b) are called the "regularity condition in the subspace of m-dimensional space" or "partial regularity condition" (see for example: in the case of quaternion variable $n=4$, we have introduced two regularity conditions: $m=2$ and $m=4$.[2])

(B) Integral theorems.

From (16a), integrating $DF(X)$ over the m-dimensional volume, we find:

(17) $$J \equiv \int_{V_m} dV_m(DF(X)) = \int_{S_{m-1}} dX\, F(X),$$

where dX is the surface element of the $m-1$-dimensional hypersurface of the m-dimensional volume V_m. In the last equation, we have used Gauss's divergence theorem over the hypersurface S_{m-1}. The dX is the surface element of the hypersurface S_{m-1} and is a hypercomplex number expressed as:

(18) $dX = \sum_{k=0}^{m-1} \xi_k i_k \, ds,$

ξ_k are the direction cosine of the normal to the surface element dX, ds is the $m-1$-dimensional volume of the surface element dX.

Let $F(X)$ be right D regular everywhere in a m-dimensional domain V_m, enclosed by a $m-1$-dimensional hypersurface S_{m-1}, then the value of the integral J defined in (17) is zero:

(19) $J = \int_{S_{m-1}} dX \, F(X) = 0.$

IV. Regular functions.

We now introduce special regular functions through a similar process we had used in the case of a quaternion variable and of an octonion variable.

Let $G(Z)$ be a function of a complex variable $z = x_0 + ix$, $i = \sqrt{-1}$, and is regular in a domain d:

(20) $G(Z) = G(x_0 + ix) = u(x_0, x) + iv(x_0, x).$

Then u, and v satisfy the following condition:

(21) $$\frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial x_0} = -\frac{\partial u}{\partial x},$$

and

$$\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x^2} \right) \binom{u}{v} = 0.$$

We now introduce a function $G(X)$ of a hypercomplex variable X.

(22) $X = x_0 + \sum_{k=1}^{m-1} i_k x_k .$

Writing X in the following form:

(23) $X = x_0 + ix,$

where

$$x = (x_1^2 + x_2^2 + \dots + x^2_{m-1})^{1/2} = a \text{ scalar},$$

$$i = x^{-1}(i_1 x_1 + i_2 x_2 + \dots + i_{m-1} x_{m-1}).$$

Replacing X in the regular function of a complex variable $G(Z)$:

(24) $$G(X) = G(x_0 + ix) = u(x_0, x) + i\, v(x_0, x)$$
$$= u(x_0, x) + \sum_{k=1}^{m-1} i_k x_k \, v(x_0, x)/x$$

Now we prove the following main theorem I.

Theorem I. Let $u^{(s)}(x_0, x)$, $v^{(s)}(x_0, x)$ and $w^{(s)}(x_0, x)$ be functions of x_0 and x and be defined as follows: (s: a positive integar),

(25) $u^{(s)}+i\,v^{(s)} \equiv \square^s F(X) \equiv \square^s(u+i\,v)$,

(26) $w^{(s)} \equiv \square^{s-1}(DF(X))$,

then, $u^{(s)}$, $v^{(s)}$ and $w^{(s)}$ are given respectively, by the following:

(27) $u^{(s)}=(m-2s)x^{-1}(u^{(s-1)})_x$, $u^{(0)}\equiv u$,

(28) $v^{(s)}=(m-2s)(x^{-1}v^{(s-1)})_x$, $v^{(0)}\equiv v$,

(29) $w^{(s)}=(m-2s)x^{-1}(w^{(s-1)})_x$, $w^{(1)}\equiv DF(X)$,

where $(\ )_x \equiv \frac{\partial}{\partial x}(\cdot)$ and $\square \equiv \sum_{k=0}^{2s-1}\frac{\partial^2}{\partial x_k^2}$.

The proof of (27), (28) and (29) are given in **Appendix.**

Theorem II. Let m be an even number: $m=2s$, and the function $G(X)$ be defined by (20) and is a function of a hypercomplex variable $X=x_0+\sum_{k=1}^{2s-1} i_k x_k$. Then

(30) $\square^s G(X)=0$,

and $F(X)$ defined by $\square^{s-1}G(X)$ is a left D regular function:

(31) $DF(X)=0$, $F(X)\equiv\square^{s-1}G(X)$.

Proof. Using equations (27) and (29), $m=2s$, we have

$$u^{(s)}+i\,v^{(s)}=(m-2s)(x^{-1}(u^{(s-1)})_x+i(x^{-1}v^{(s-1)})_x)=0.$$

(32) $\square^s G(X)=0$.

Also, from (29),

(33) $w^{(s)}=\square^{s-1}DG(X)=(m-2s)(x^{-1}w_x^{(s-1)})=0$.

Since m can be any even number not greater than n, we have $[n/2]=p$ different regularity conditions by taking the variable X as $X=x_0+\sum_{k=1}^{s-1} i_k x_k$, $s=1, 2, \dots, [n/2]=p$, where $[a]$ is the integrar part of a.

For each s, we have $(n-1)!/(2s-1)!(n-2s)!$ ways of choosing $i_{k_1}, i_{k_2}, \dots, i_{k_{2s-1}}$ from $(i_1, i_2, \dots, i_n)$.

For example, in the case of functions of an octonion variable we have, using a regular function of a complex variable $G(X)$, the following regularity conditions:

(34) $D_m F(X_m)=0$,

where

$$F(X_m)=\square^{s-1}G(X_m),\quad m=2s,$$

for $m=2s=2, 4, 6, 8.$,

$$D_m \equiv \frac{\partial}{\partial x_0}+\sum_{k=1}^{m-1} i_k\frac{\partial}{\partial x_k},$$

$$\Box_m \equiv \frac{\partial^2}{\partial x_0^2} + \sum_{k=1}^{m-1} \frac{\partial^2}{\partial x_k^2}\,.$$

When $s=1$, $X_2 = x_0 + ix$ which is the same variable as the complex variable. When $s=2$, $X_4 = x_0 + i_1x_1 + i_2x_2 + i_3x_3$ which is equivalent to the quaternion variable we had discussed before.

When $s=3$, we have a regualrity condition in the six dimensional space in the 8-dimensional octonion space:

$$D_6\, F(x_0 + i_1x_1 + \dots + i_5x_5) = 0.$$

We introduce another type of regular functions using generating functions:

(35) $$K^n(X, \vec{t}\,) \equiv (x_0\vec{t} + (\vec{x}\cdot\vec{t}\,))^n = (\sum_{k=1}^{m-1} t_k(x_k + i_kx_0))^n$$

where $\vec{t} = \sum_{k=1}^{m-1} i_kt_k$, $\vec{x} = \sum_{k=1}^{m-1} i_kx_k$, $(\vec{x}\cdot\vec{t}\,) = \sum_{k=1}^{m-1} x_kt_k$.

We define polynomials $P_{n_1n_2\cdots n_{m-1}}(X)$ as follows:

(36) $$K^n(X, \vec{t}\,) = \sum_{(n_1,\dots,n_{m-1})}^{\Sigma n_k = n} n!\; P_{n_1,\dots,n_{m-1}}(X)\; t_1^{n_1} t_2^{n_2} \dots t_{m-1}^{n_{m-1}}.$$

We show that $P_{n_1\cdots n_{m-1}}(X)$ is both side $\bar{D}$ regular, where $\bar{D}$ is defined as

$$\bar{D} \equiv \frac{\partial}{\partial x_0} - \sum_{k=1}^{m-1} i_k \frac{\partial}{\partial x_k}\,.$$

Proof. We have:

$$\begin{aligned}
\bar{D}\,K^n &= \left(\frac{\partial}{\partial x_0} - \sum_{k=1}^{m-1} i_k \frac{\partial}{\partial x_k}\right) K^n \\
&= \sum_{r=0}^{m-1} K^{n-r}\vec{t}\, K^{r-1} - \sum_{k=1}^{m-1}\sum_{r=0}^{n-1} i_k K^{n-r} t_k\, K^{r-1} \\
&= \sum_{r=0}^{n-1} \vec{t}\, K^{n-1} - \sum_{r=0}^{n-1} \vec{t}\, K^{n-1}.
\end{aligned}$$

Because $\vec{t}$ and K commute: $\vec{t}\,K = K\vec{t}$, the right-hand side of the above equation is zero.

(37) $$\bar{D}K^n = 0.$$

That is K^n is $\bar{D}$ left regular and we can show also that K^n is right $\bar{D}$ regular by the same procedure as the above.

Now we derive both side $\bar{D}$ regular polynomials of x_k. Expanding K^n in a polynomial of $t_1, t_2, \dots, t_{m-1}$:

$$\begin{aligned}
K^n &= (t_1(x_1 + i_1x_0) + \dots + t_{m-1}(x_{m-1} + i_{m-1}x_0))^n \\
&= \sum_{(n_1,n_2,\dots,n_{m-1})}^{\Sigma n_k = n} n!\; P_{n_1n_2\cdots n_{m-1}}(X)\; t_1^{n_1} t_2^{n_2} \dots t_{m-1}^{n_{m-1}}
\end{aligned}$$

where the summation is taken for all combination of $(n_1, n_2, \dots, n_{m-1})$ subject to the restriction: $\sum_{k=1}^{m-1} n_k = n$.

Differentiating K^n with respect to $t_1, t_2, \dots t_{m-1}$, respectively, $n_1, n_2, \dots, n_{m-1}$ times, we find from (36):

(37a) $$\frac{1}{n_1!\, n_2! \dots n_{m-1}!} \frac{\partial^n K^n}{\partial t_1^{n_1} \partial t_2^{n_2} \dots \partial t_{m-1}^{n_{m-1}}} = n!\, P_{n_1 n_2 \cdots n_{m-1}}(X),$$

where $\sum_{k=1}^{m-1} r_k = n$.

(37b) $$\bar{D}P_{n_1 n_2 \cdots n_{m-1}}(X) = \frac{1}{n!} \frac{1}{\prod_{r=1}^{m-1} n_r!} \frac{\partial^n}{\partial t_1^{n_1} \dots \partial t_{m-1}^{n_{m-1}}} \bar{D}K^n = 0.$$

Thus, $P_{n_1 n_2 \cdots n_{m-1}}(X)$ is left $\bar{D}$ regular.

Some relations among P functions are described in the following.

Let $\sum_{k=1}^{m-1} r_k = \sum_{k=1}^{m-1} n_k = n$, then

(38) $$\frac{\partial^n}{\partial x_1^{r_1} \partial x_2^{r_2} \dots \partial x_{m-1}^{r_{m-1}}} P_{n_1 n_2 \cdots n_{m-1}}(X) = \delta_{n_1 r_1} \delta_{n_2 r_2} \dots \delta_{n_{m-1} r_{m-1}},$$

(39) $$\frac{\partial P_{n_1 n_2 \cdots n_{m-1}}(X)}{\partial x_l} = P_{n_1 \cdots n_l - 1 \cdots n_{m-1}}(X).$$

To prove the above relations we prove first (39).

Proof: Differentiate K^n with respect to x_l:

$$\frac{\partial K^n}{\partial x_l} = \sum_{r=0}^{n-1} K^{n-r} t_l K^{r-1} = n t_l K^{n-1}$$

$$= n t_l \sum_{(n \cdots n_{m-1})}^{(\Sigma n_k = n-1)} (n-1)!\, P_{n_1 \cdots n_l \cdots n_{m-1}}(X)\, t_1^{n_1} \cdots t_l^{n_l} \cdots \dots t_{m-1}^{n_{m-1}}$$

$$= \Sigma\, n!\, P_{n_1 \cdots n_l \cdots n_{m-1}}(X)\, t_1^{n_1} \cdots t_l^{n_l+1} \dots t_{m-1}^{n_{m-1}}$$

(40) $$= \Sigma\, n!\, P_{n_1 \cdots n'_l - 1 \cdots n_{m-1}}(X)\, t_1^{n_1} \cdots t_l^{n'_l} \cdots t_{m-1}^{n_{m-1}}.$$

Using (36) for K^n, we have

(41) $$\frac{\partial K^n}{\partial x_l} = \sum_{(n_1, \dots, n_{m-1})} n! \left(\frac{\partial}{\partial x_l} P_{n_1 \cdots n_l \cdots n_{m-1}}(X) \right) t_1^{n_1} \dots t_l^{n_l} \dots t_{m-1}^{n_{m-1}},$$

Comparing the coefficients of $t_k^{n_k}$ of (40) and (41), we have

(39) $$\frac{\partial}{\partial x_l} P_{n_1 \cdots n_l \cdots n_{m-1}}(X) = P_{n_1 \cdots n_l - 1 \cdots n_{m-1}}(X).$$

By repeating the process (39), we obtain:

(38) $$\frac{\partial^n}{\partial x_1^{n_1} \cdots \partial x_{m-1}^{n_{m-1}}} P_{n_1 \cdots n_{m-1}}(X) = P_{00 \cdots 0}(X) = 1$$

and from

$$\frac{\partial^n (x_0 \vec{t} + (\vec{x} \cdot \vec{t}))^n}{\partial x_1^{n_1} \cdots \partial x_{m-1}^{n_{m-1}}} = n!\,(t_1^{n_1} \cdots t_{m-1}^{n_{m-1}}) = \Sigma\, n! \left(\frac{\partial^n P_{n_1 \cdots n_{m-1}}(X)}{\partial x_1^{n_1} \cdots \partial x_{m-1}^{n_{m-1}}} \right) t_1^{n_1} \cdots t_{m-1}^{n_{m-1}}$$

we obtain (38).

V. Fourier expansion.

(A) Exponential functions.

Let us define an exponential function as follows:

(42) $$e^{i(x_0\vec{t}+(\vec{t}\cdot\vec{x}))}=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}(x_0\vec{t}+(\vec{x}\cdot\vec{t}))^n=\sum_{n=0}^{\infty}\frac{(i)^n}{n!}K^n$$

The exponential function is both side $\bar{D}$ regular:

(43) $$\bar{D}\,e^{i(x_0\vec{t}+(\vec{x}\cdot\vec{t}))}=e^{i(x_0\vec{t}+(\vec{x}\cdot\vec{t}))}\bar{D}=0.$$

Since $(\vec{x}\cdot\vec{t})$ is scalar:

(44) $$e^{i(x_0\vec{t}+(\vec{x}\cdot\vec{t}))}=e^{i(\vec{x}\cdot\vec{t})}e^{ix_0\vec{t}}$$

$$=e^{i(\vec{x}\cdot\vec{t})}\left[\cos(x_0t)+\frac{i\vec{t}}{t}\sin(x_0t)\right]$$

where $t=(\sum_{k=1}^{n-1}t_k^2)^{\frac{1}{2}}$.

(B) Fourier representation of regular functions.

Let us consider the following function:

(45) $$\Phi(X)=\int_{-\infty}^{+\infty}\cdots\int e^{i(x_0\vec{t}+(\vec{x}\cdot\vec{t}))}A(\vec{t})\,dt_1dt_2\cdots dt_{m-1}$$

where $A(\vec{t})$ is a function of $t_1, t_2, \cdots, t_{m-1}$.

$\Phi(X)$ is left $\bar{D}$ regular:

(46) $$\bar{D}\Phi(X)=0.$$

Putting $x_0=0$ in $\Phi(X)$, we have

(47) $$\Phi(X)|_{x_0=0}=\Phi(\vec{x})=\int\cdots\int e^{i(\vec{x}\cdot\vec{t})}A(\vec{t})\,dt_1dt_2\cdots dt_{m-1}$$

which is a Fourier integral. $A(\vec{t})$, then is expressed as

(48) $$A(\vec{t})=\frac{1}{(2\pi)^{m-1}}\int_{-\infty}^{+\infty}\cdots\int e^{-i(\vec{t}\cdot\vec{\tau})}\Phi(\vec{\tau})\,d\tau_1d\tau_2\cdots d\tau_{m-1}$$

Inserting $A(\vec{t})$ in $\Phi(X)$, we have

(49) $$\Phi(X)=\frac{1}{(2\pi)^{m-1}}\int_{-\infty}^{+\infty}\cdots\int e^{i(x_0\vec{t}+\vec{t}(\vec{\tau}-\vec{x}))}\Phi(\vec{\tau})\,d\tau_1d\tau_2\cdots d\tau_{m-1}$$

Appendix. Proof of theorem I.

From the equation (23) and the following two equations, we have:

(A-1) $$\frac{\partial}{\partial x_k}i=\frac{\partial}{\partial x_k}(x^{-1}(\sum_{j=1}^{m-1}i_jx_j))=i_kx^{-1}-i\,x_kx^{-2},$$

(A-2) $$\sum_{k=1}^{m-1}\left(\frac{\partial i}{\partial x_k}\right)\left(\frac{\partial f(x_0,x)}{\partial x_k}\right)=0,$$

(A-3) $$\sum_{k=1}^{m-1}\frac{\partial^2}{\partial x_k^2}i=-(m-2)i\,x^{-2},$$

(A-4) $\sum_{k=1}^{m-1} \frac{\partial^2}{\partial x_k^2} f(x_0, x)=(m-2)x^{-1}(f(x_0, x))_x+(f(x_0, x))_{xx}$,

where $(\ )_{xx}\equiv \frac{\partial^2}{\partial x^2}(\ )$ and $f(x_0, x)$ is a function of x_0 and x.

From (A-1) through to (A-4), we have

(A-5) $\Box f(x_0, x)=f_{00}+f_{xx}+(m-2)x^{-1}(f)_x$,

(A-6) $\Box (if(x_0, x))=i(f_{00}+f_{xx}+(m-2)(x^{-1}f)_x)$,

(A-7) $D(if(x_0, x))=-f_x+if_0-(m-2)x^{-1}f$.

Using the relations (21), we find from (A-5) through to (A-7):

(A-8) $\Box (u+iv)\equiv u^{(1)}+iv^{(1)}=(m-2)(x^{-1}u_x+i(x^{-1}v)_x)$,

(A-9) $D(u+iv) = -(m-2)x^{-1}v\equiv w^{(1)}$.

In the above equations, we have used the relations given by (21).

We now prove the following equations:

(A-10) $u^{(r)}=\Box^r u(x_0, x)=(m-2r)(x^{-1}u_x{}^{(r-1)})$,

(A-11) $iv^{(r)}=\Box^r (iv(x_0, x))=i(m-2r)(x^{-1}v^{(r-1)})_x$,

(A-12) $w^{(r)}=\Box^{r-1}(D(u(x_0, x)+iv(x_0, x)))=(m-2r)(x^{-1}w_x{}^{(r-1)})$

and

(A-13) $u_{00}^{(r-1)}+u_{xx}^{(r-1)} = -2(r-1)x^{-1}u_x^{(r-1)}$,

(A-14) $v_{00}^{(r-1)}+v_{xx}^{(r-1)} = -2(r-1)(x^{-1}v^{(r-1)})_x$,

(A-15) $w_{00}^{(r-1)}+w_{xx}^{(r-1)} = -2(r-1)x^{-1}w_x^{(r-1)}$.

We see that by equations (A-8), (A-9) and (21) and identifying $u^{(0)}\equiv u$, $v^{(0)}\equiv v$, and $w^{(1)}\equiv D(u+iV)$, equations (A-10) through to (A-15) hold true for $r=1$. Next, assuming equations (A-10) through to (A-15) hold good for $r=r$, we prove those equations hold good also for $r=r+1$.

(A) First, we prove (A-13) for $r=r+1$.

Take $u^{(r)}$ and use equation (A-10) for $r=r$, we have:

$$I\equiv u_{00}^{(r)}+u_{xx}^{(r)}\equiv(m-2r)((x^{-1}u_x^{(r-1)})_{00}+(x^{-1}u_x^{(r-1)})_{xx})$$

Since x does not depend on x_0, we obtain the following:

$$I=(m-2r)(x^{-1}(u_{00}^{(r-1)}+u_{xx}^{(r-1)})_x-2x^{-1}(x^{-1}u_x^{(r-1)})_x),$$

Since (A-13) holds good for $r=r$, applying (A-13) for $r=r$ to the above equation, we have

(A-16) $\mathrm{I}\equiv u_{00}^{(r)}+u_{xx}^{(r)} = -2r(m-2r)(x^{-1}(x^{-1}u_x^{(r-1)})_x)$.

Using (A-10) for $r=r$, we have $I=-2r(x^{-1}u_x{}^{(r)})$ which gives equation (A-13) for $r=r+1$. Thus. (A-13) holds good for all positive integars r.

(B) We now prove (A-10) for $r=r+1$, provided that (A-10) and (A-13) hold good for $r=r$.

Using the relation (A-5) for $u^{(r)}$, we have:

$$u^{(r+1)} \equiv \Box u^{(r)} = u^{(r)}_{00} + u^{(r)}_{xx} + (m-2)(x^{-1}u^{(r)}_{x})$$

Applying the relation obtained in (A): (A-16), we have

$$u^{(r+1)} \equiv (m-2(r+1))(x^{-1}u^{(r)}_{x}).$$

Thus, we have proved (A-10) for $r=r+1$.

(C) Assuming (A-14) and (A-11) hold good for $r=r$ and prove they hold good for $r=r+1$.

Let I_{r+1} be

$$I_{r+1} \equiv v^{(r)}_{00} + v^{(r)}_{xx} = (m-2r)\,((x^{-1}v^{(r-1)})_{x00} + (x^{-1}v^{(r-1)})_{xxx})$$

$$= (m-2r)(x^{-1}(v^{(r-1)}_{00} + v^{(r-1)}_{xx}) - 2x^{-1}(x^{-1}v^{(r-1)})_{x})_{x}$$

By the assumption that (A-14) holds good for $r=r$, we obtain using (A-14):

(A-17) $\quad I_{r+1} = (m-2r)(-2r)(x^{-1}(x^{-1}v^{(r-1)})_{x})_{x} = -2r(x^{-1}v^{(r)})_{x}$.

In the last equation (A-11) for $r=r$ is used. (A-17) is just the equation (A-14) for $r=r+1$. Thus, (A-14) holds good for all positive integars r.

(D) Now we prove (A-11) for $r=r+1$, provided that (A-17) holds good. Using (A-6) for $f=v^{(r+1)}(x_0, x)$, we have:

$$v^{(r+1)} = -i\Box v^{(r)} = (v^{(r)}_{00} + v^{(r)}_{xx} + (n-2)(x^{-1}v^{(r)})_{x}).$$

Applying eq. (A-17) for $v^{(r)}_{00} + v^{(r)}_{xx} \equiv I_{r+1}$, we have equation (A-11) for $r=r+1$. Thus, we have proved.

(E) (A-12) and (A-15) can be proved by replacing $u^{(r)}$ by $w^{(1)} = D(u+iv)$ in (A-10) and (A-13), respectively.

Conclusion.

We have seen that many of the general method and results can be extended to the case of a higher hypercomplex variable such as an octonion variable, a 16-nion variable and so on for which we do not assume the associative law nor the alternative law but assume the power associative law for constructing regular functions.

Thus, the general and common features of the theory of functions of a hypercomplex variable are largely dependent on the power associative laws. The specific features of the theory of functions of a particular hypercomplex variable is not yet established in the present paper and has to wait for some time to come.

References:

(1) K. Imaeda and Mari Imaeda: The Bulletin of the Okayama Univ. of Science, No. **19A**, 93 (1984), P. Dentoni and M. Sce: Rend. Sem. Math., Univ. Padova **50**, 251 (1973)

(2) K. Imaeda: Nuovo Cimento: **32B**, 138 (1976), Report-FPL-183-1, Faculty of Science, Okayama Univ. of Science (1983).

(3) R.D. Schafer: An Introduction to Nonassociative Algebras, Academic Press 1966.

(4) C.W. Curtis: The Four aud Eight Square Problem and Division Algebras., Studies in Modern Algebra, Ed. A.A. Albert. Published by Prentice-Hall, Inc..