GRAVITY INDUCED BY THE LOCAL SL(2, C) TRANSFORMATION

Shin-ich NAKARIKI

Department of Applied Physics, Okayama University of Science
Ridai-cho 1-1 Okayama, 700 Japan

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ABSTRACT

It is insisted that the local SL(2, C) transformation of a spin basis $\boldsymbol{\omega}_A$ generates not only the local $SO_0(3, 1)$ transformation of a Lorentz basis $\mathbf{E}_{\underline{\mu}}$, but also the extended GL (4, R) transformation of a coordinate basis ∂_μ.

The two bases $\mathbf{E}_{\underline{\mu}}$ and ∂_μ are chosen, at the outset, so as to coincide with each other in the flat spacetime. The separation of two bases $\mathbf{E}_{\underline{\mu}}$ and ∂_μ suggests at once the existence of a field $h_\mu{}^\nu$ which connects them. Using this field it is shown that the invariant action under the local SL(2, C) transformation leads to the Einstein's theory of gravity including the extension to the case that a matter field q is a spinor, provided that the original action is invariant under the Poincaré transformation.

§ 1. INTRODUCTION

Since Utiyama[1] pointed out in 1956 that a gravitational field was also a kind of gauge fields, the extensive works have been done on this subject.

These works could be classified into the following three categories in terms of different gauge groups, i.e.,

(A) Lorentz[1] or SL(2, C)[2),3] gauge theory,

(B) translational gauge theory[4),5],

(C) Poincaré gauge theory[6].

To begin, we review these gauge theories from the view-point of the gauge group operations.

First, we give the definition of terms used in the following. We shall call the gauge theory under consideration the external (or briefly E), internal (I), or external-internal (E-I) theory, in case corresponding respectively to the case that the generalized transformations of the group cause the transformation of coordinates, fields, or both of them. It is obvious that the theory of (C) type is an E-I theory. However, the theory of (B) type may be either E-theory[4] or E-I theory[5] looking upon the generalized translations, i.e., the general transformation of coordinates as the generalization of the translations or the translations plus Lorentz transformations respectively. Next, let us consider that the theory of (A) type could correspond to which of the E, I, or E-I theory defined above.

First of all, it should be noticed that we have two viewpoint[7] on behavior of the coefficient matrices $\boldsymbol{\sigma}^{\mu}$ ($\mu=0, 1, 2, 3$) of Dirac equations under the transformation of coordinates, i.e.,

(I) $\boldsymbol{\sigma}^{\mu}$ is an invariant constant matrix,

(II) $\boldsymbol{\sigma}^{\mu}$ is transformed like a contravariant vector.

In the flat spacetime background it seems to be natural to adopt the viewpoint (I). This viewpoint gives the connection of the external transformation of coordinates with the internal transformation of spinors.

However, if we must consider the general transformation of coordinates in the curved spacetime background, then this connection will be broken, and it will be required that $\boldsymbol{\sigma}^{\mu}$ should be taken as a contravariant vector,[8] because there are not the spinor transformations corresponding to the general transformation of coordinates.

The conventional (A) type gauge theories of the gravity have been formulated as the invariant I-theories under the general transformation of coordinates in the curved spacetime background, regarding $\boldsymbol{\sigma}^{\mu}$ as a mixed quantity of vector and spinor[2] or by introducing the fields equivalent to $\boldsymbol{\sigma}^{\mu}$'s[1),3)] in order to separate the internal transformations from the external ones. However, we will think here that the connection of external and internal transformations is maintained also under the local SL(2, C) transformations. Then it will not be necessary to regard *a priori* $\boldsymbol{\sigma}^{\mu}$ as a mixed quantity of vector and spinor. Perhaps then, the gauge theory of this type could be formulated as E-I theory in the flat spacetime background. The purpose of this paper is to explore this possibility and show that it just leads to the Einstein's theory of gravity including the extension to the case that a matter field is a spinor.

The presentation is divided into four parts.

In § 2, we briefly review the concepts of a Lorentz, coordinate and spin basis, respectively and the relations among them in a general spacetime manifold. In § 3, we consider the local SL(2, C) spin transformations and discuss the Lorentz-and coordinate-basis transformations induced by them. There, the field $h_{\underline{\mu}}{}^{\nu}$ will be introduced. In § 4, the invariant action under the local SL(2, C) transformations will be presented in the viewpoint mentioned in § 3 and the final section is devoted to conclusion.

§ 2 LORENTZ, COORDINATE AND SPIN BASIS

We assume that the spacetime continuum M is a Hausdorff, connected C^∞ four-dimensional manifold with affine connection and on which a local Minkowskian structure is defined. It is well known that the local Minkowskian structure allows us to introduce a spinor structure on M.

Let Tp be a tangent vector space at a point P on M and Tp* be its dual space. And let $(S_2)_p$, $(S^*_2)_p$, $(\bar{S}_2)_p$ and $(\bar{S}^*_2)_p$ be a two-dimensional symplectic spinor space, its dual space and their complex-conjugate spaces respectively. We can adopt such the bases $\mathbf{E}_{\underline{\mu}}$ $[\mathbf{E}^{\underline{\mu}}]$ ($\mu=0, 1, 2, 3$) (called Lorentz bases) of Tp [Tp*] as $\mathbf{g}(\mathbf{E}^{\underline{\mu}}, \mathbf{E}_{\underline{\nu}})=\eta_{\mu\nu}$, $\mathbf{g}^{-1}(\mathbf{E}^{\underline{\mu}}, \mathbf{E}^{\underline{\nu}})=\eta^{\mu\nu}$, where $(\eta_{\mu\nu})=(\eta^{\mu\nu})=\mathrm{diag.}\ (1, -1, -1, -1)$, besides the coordinate bases*) $\mathbf{e}_\mu\equiv(\partial_\mu)_p$ $[\mathbf{e}^\mu\equiv(dx^\mu)_p]$ for which $\mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)=g_{\mu\nu}$, $\mathbf{g}^{-1}(\mathbf{e}^\mu, \mathbf{e}^\nu)=g^{\mu\nu}$: $g_{\mu\nu}g^{\nu\lambda}=\delta^\lambda_\mu$. The $\mathbf{g}$ is a metric tensor defined over M and defines at P an inner product $\mathbf{g}(\mathbf{u}, \mathbf{v})=\eta_{\mu\nu}u^{\underline{\mu}}v^{\underline{\nu}}=g_{\mu\nu}u^\mu v^\nu$ for any $\mathbf{u}, \mathbf{v}\in$ Tp, where $u^{\underline{\mu}}$ and u^μ are the components of $\mathbf{u}$ with respect to the bases $\mathbf{E}_{\underline{\mu}}$ and $\mathbf{e}_\mu$ respectively, i.e., $\mathbf{u}=u^{\underline{\mu}}\mathbf{E}_{\underline{\mu}}=u^\nu\mathbf{e}_\nu$. The $\mathbf{g}$ gives also a linear isomorphism of Tp onto Tp*, i.e., $\mathbf{g}$: $\mathbf{u}\to\mathbf{u}^*=\mathbf{g}(\mathbf{u},\ \ )\in$ Tp*, where $\mathbf{u}^*=u_{\underline{\mu}}\mathbf{E}^{\underline{\mu}}=u_\mu\mathbf{e}^\mu$; $u_{\underline{\mu}}=\eta_{\mu\nu}u^{\underline{\nu}}$, $u_\mu=g_{\mu\nu}u^\nu$.

Then the $\mathbf{g}^{-1}$ is the inverse of this linear isomorphism $\mathbf{g}$ and gives also a bilinear map of Tp*×Tp* into R, i.e., if $\mathbf{g}^{-1}$: $\mathbf{u}^*, \mathbf{v}^*\to\mathbf{u}=\mathbf{g}^{-1}(\mathbf{u}^*,\ \ )$, $\mathbf{v}=\mathbf{g}^{-1}(\mathbf{v}^*,\ \ )$, then $\mathbf{g}^{-1}(\mathbf{u}^*, \mathbf{v}^*)=\eta^{\mu\nu}u_{\underline{\mu}}v_{\underline{\nu}}=g^{\mu\nu}u_\mu v_\nu=\mathbf{g}(\mathbf{u}, \mathbf{v})$.

On the other hand we can always choose such the bases $\boldsymbol{\omega}_A$ $[\boldsymbol{\omega}^A]$ ($A=0, 1$) of $(S_2)_p$ $[(S^*_2)_p]$ as $\varepsilon(\boldsymbol{\omega}_A, \boldsymbol{\omega}_B)=\varepsilon_{AB}$, $\varepsilon^{-1}(\boldsymbol{\omega}^A, \boldsymbol{\omega}^B)=\varepsilon^{AB}$, $\varepsilon_{AB}=\delta_A{}^0\delta_B{}^1-\delta_A{}^1\delta_B{}^0$, $\varepsilon^{AB}=\delta_0{}^A\delta_1{}^B-\delta_1{}^A\delta_0{}^B$; $\varepsilon_{AB}\varepsilon^{AC}=\delta^C_B$. The ε is a spinor metric and defines at P an inner product $\varepsilon(\boldsymbol{\phi}, \boldsymbol{\psi})=\varepsilon_{AB}\phi^A\psi^B$ for any $\boldsymbol{\phi},\boldsymbol{\psi}\in$ Sp, where $\boldsymbol{\phi}=\phi^A\boldsymbol{\omega}_A$.

The ε gives also a linear isomorphism of $(S_2)_p$ into $(S^*_2)_p$, i.e., ε: $\boldsymbol{\phi}\to\boldsymbol{\phi}^*=\varepsilon(\boldsymbol{\phi},\ \ )$, where $\boldsymbol{\phi}^*=\phi_A\boldsymbol{\omega}^A$; $\phi_A=\phi^B\varepsilon_{BA}$. The ε^{-1} is then the inverse of ε and gives also a bilinear map of $(S^*_2)_p\times(S^*_2)_p$ into C, i.e., $\varepsilon^{-1}(\boldsymbol{\phi}^*, \boldsymbol{\psi}^*)=\varepsilon^{AB}\phi_A\psi_B=\varepsilon_{AB}\phi^A\psi^B=$

* Notice that the existence of a local coordinate neighbourhood U with the local coordinates x^μ ($\mu=0, 1, 2, 3$) is assumed at each point on M.

$\varepsilon(\phi, \psi)$.

The complex-conjugates $\bar{\varepsilon}=\varepsilon_{\dot{A}\dot{B}}\boldsymbol{\omega}^{\dot{A}}\otimes\boldsymbol{\omega}^{\dot{B}}$ and $\bar{\varepsilon}^{-1}=\varepsilon^{\dot{A}\dot{B}}\boldsymbol{\omega}_{\dot{A}}\otimes\boldsymbol{\omega}_{\dot{B}}$ play the same role in the complex-conjugate spaces $(\bar{S}_2)_p$ and $(\bar{S}^*_2)_p$ respectively. Here and in the following, we fix that the dotted indices denote the complex conjugates of the corresponding undotted quantities.

We can construct the tensor product spaces of any types from the spaces Tp, Tp*; $(S_2)_p$, $(S^*_2)_p$ and their conjugates;

$$\mathbf{T}^{r:\ t\ \dot{v}}_{s:\ u\ \dot{w}}=Tp^{\otimes r}\otimes Tp^{*\otimes s}(S_2)_p{}^{\otimes t}(S^*_2)_p{}^{\otimes u}\otimes(\bar{S}_2)_p{}^{\otimes v}\otimes(\bar{S}^*_2)_p{}^{\otimes w},$$

where $Tp^{\otimes r}$ denotes $\underbrace{Tp\otimes\cdots\otimes Tp}_{r\ \text{factors}}$.

We call an element of $\mathbf{T}^{r:\ t\ \dot{v}}_{s:\ u\ \dot{w}}$ a tensor-spinor of type $(r, s;\ t, u;\ \dot{v}, \dot{w})$.

It is well known that the tangent vector space Tp is topologically isomorphic to the Hermitian product space $(\bar{S}_2)_p\otimes_H(S_2)_p$.

We shall treat them as being identical in the following.

Thereby we shall be able to discuss a tensor-spinor in terms of its equivalent spinor. We should stress here that the basis of Tp which is represented by it of $(\bar{S}_2)_p\otimes_H(S_2)_p$ is a Lorentz basis, i.e.,

$$\mathbf{E}_{\underline{\mu}}=\sigma_{\underline{\mu}}{}^{\dot{A}B}\boldsymbol{\omega}_{\dot{A}}\otimes\boldsymbol{\omega}_B\,, \tag{2.1}$$

where $\boldsymbol{\sigma}_{\underline{\mu}}$'s equal to the constant coefficient matrices of Dirac equations and satisfy the relations

$$\sigma_{\underline{\mu}}{}^{\dot{A}B}\sigma_{\underline{\nu}\dot{A}C}+\sigma_{\underline{\nu}}{}^{\dot{A}B}\sigma_{\underline{\mu}\dot{A}C}=\eta_{\mu\nu}\delta^B_C. \tag{2.2}$$

On the other hand the relations between the other basis of Tp, such as a coordinate basis, and the basis of $(\bar{S}_2)_p\otimes_H(S_2)_p$ are given by

$$\mathbf{e}_\mu=\sigma\check{}_\mu{}^{\dot{A}B}\boldsymbol{\omega}_{\dot{A}}\otimes\boldsymbol{\omega}_B\,. \tag{2.3}$$

Here the matrix $\boldsymbol{\sigma}\check{}_\mu$ is related to $\boldsymbol{\sigma}_{\underline{\mu}}$ by

$$\boldsymbol{\sigma}\check{}_\mu=h^{\underline{\nu}}{}_\mu\boldsymbol{\sigma}_{\underline{\nu}}\,,\qquad \boldsymbol{\sigma}_{\underline{\mu}}=h_{\underline{\mu}}{}^\nu\boldsymbol{\sigma}\check{}_\nu \tag{2.4}$$

since the Lorentz basis $\mathbf{E}_{\underline{\mu}}$ is connected with the coordinate basis $\mathbf{e}_\mu$ through "a field" $h_{\underline{\mu}}{}^\nu$ (or $h^{\underline{\nu}}{}_\mu$);

$$\mathbf{E}_{\underline{\mu}}=h_{\underline{\mu}}{}^\nu\mathbf{e}_\nu\,,\qquad \mathbf{e}_\mu=h^{\underline{\nu}}{}_\mu\mathbf{E}_{\underline{\nu}}\,. \tag{2.5}$$

Owing to the eq. (2.2) we see then that $\boldsymbol{\sigma}\check{}_\mu$'s satisfy the relations

$$\sigma\check{}_\mu{}^{\dot{A}B}\sigma\check{}_{\nu\dot{A}C}+\sigma\check{}_\nu{}^{\dot{A}B}\sigma\check{}_{\mu\dot{A}C}=g_{\mu\nu}\delta^B_C\,, \tag{2.6}$$

where $g_{\mu\nu}=\mathbf{g}(\mathbf{e}_\mu, \mathbf{e}_\nu)=h^{\underline{\lambda}}{}_\mu h^{\underline{\kappa}}{}_\nu\mathbf{g}(\mathbf{E}_{\underline{\lambda}}, \mathbf{E}_{\underline{\kappa}})=h^{\underline{\lambda}}{}_\mu h^{\underline{\kappa}}{}_\nu\eta_{\lambda\kappa}=h^{\underline{\lambda}}{}_\mu h_{\underline{\lambda}\nu}$.

We notice here the following.

Owing to the relation (2.1) we can find the corresponding Lorentz basis whenever we treat the SL(2, C) spin-basis transformation. Conversely, we can always find the corresponding spin basis as long as we treat the only Lorentz-basis transformation which satisfies $\mathbf{g}(\mathbf{E}'_{\underline{\mu}}, \mathbf{E}'_{\underline{\nu}})=\mathbf{g}(\mathbf{E}_{\underline{\mu}}, \mathbf{E}_{\underline{\nu}})$, i.e., a Lorentz transformation. However, we can not find the spin-basis transformations corresponding to the general transformations of a Lorentz basis. In this case the Lorentz basis is shifted to one of coordinate bases. As a result, the relation (2.1) is broken and turned to the relation (2.3).

There, $\boldsymbol{\sigma}^{\frown}{}_{\mu}$'s are no longer constants and transformed under the spin- and/or coordinate-basis transformations.

Now, we are in position to write down any tensor-spinor of type $(r, s;\ t, u;\ \dot v, \dot w)$ in terms of the equivalent spinor;

$$\mathrm{Tp}=\mathrm{T}^{\dot A B \dot C D\cdots}{}_{\dot E F \dot G H\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\underbrace{\boldsymbol{\omega}_{\dot A}\otimes\boldsymbol{\omega}_{B}\otimes\boldsymbol{\omega}_{\dot C}\otimes\boldsymbol{\omega}_{D}\otimes\cdots}_{2r\ \text{factors}}\otimes\underbrace{\boldsymbol{\omega}^{\dot E}\otimes\boldsymbol{\omega}^{F}\otimes\boldsymbol{\omega}^{\dot G}\otimes\boldsymbol{\omega}^{H}\otimes\cdots}_{2s\ \text{factors}}$$

$$\underbrace{\otimes\boldsymbol{\omega}_{J}\otimes\boldsymbol{\omega}_{K}\otimes\cdots}_{t\ \text{factors}}\otimes\underbrace{\boldsymbol{\omega}^{L}\otimes\boldsymbol{\omega}^{M}\otimes\cdots}_{u\ \text{factors}}\otimes\underbrace{\boldsymbol{\omega}_{\dot N}\otimes\boldsymbol{\omega}_{\dot Q}\otimes\cdots}_{v\ \text{factors}}\otimes\underbrace{\boldsymbol{\omega}^{\dot R}\otimes\boldsymbol{\omega}^{\dot S}\otimes\cdots}_{w\ \text{factors}}, \tag{2.7}$$

where

$$\begin{aligned}&\mathrm{T}^{\dot A B \dot C D\cdots}{}_{\dot E F \dot G H\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\\ &=\mathrm{T}^{\underline{\mu}\,\underline{\nu}\cdots}{}_{\underline{\lambda}\,\underline{\kappa}\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\sigma_{\underline{\mu}}{}^{\dot A B}\sigma_{\underline{\nu}}{}^{\dot C D}\cdots\sigma^{\underline{\lambda}}{}_{\dot E F}\sigma^{\underline{\kappa}}{}_{\dot G H}\cdots\end{aligned} \tag{2.8}$$

or

$$=\mathrm{T}^{\mu\nu\cdots}{}_{\lambda\kappa\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\sigma^{\frown}{}_{\mu}{}^{\dot A B}\sigma^{\frown}{}_{\nu}{}^{\dot C D}\cdots\sigma^{\frown\lambda}{}_{\dot E F}\sigma^{\frown\kappa}{}_{\dot G H}\cdots$$

and conversely

$$\begin{aligned}&\mathrm{T}^{\underline{\mu}\,\underline{\nu}\cdots}{}_{\underline{\lambda}\,\underline{\kappa}\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\\ &=\mathrm{T}^{\dot A B \dot C D\cdots}{}_{\dot E F \dot G H\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\sigma^{\underline{\mu}}{}_{\dot A B}\sigma^{\underline{\nu}}{}_{\dot C D}\cdots\sigma_{\underline{\lambda}}{}^{\dot E F}\sigma_{\underline{\kappa}}{}^{\dot G H}\cdots\end{aligned}$$

or

$$\begin{aligned}&\mathrm{T}^{\mu\nu\cdots}{}_{\lambda\kappa\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\\ &=\mathrm{T}^{\dot A B \dot C D\cdots}{}_{\dot E F \dot G H\cdots}{}^{JK\cdots}{}_{LM\cdots}{}^{\dot N \dot Q\cdots}{}_{\dot R \dot S\cdots}\sigma^{\frown\mu}{}_{\dot A B}\sigma^{\frown\nu}{}_{\dot C D}\cdots\sigma^{\frown}{}_{\lambda}{}^{\dot E F}\sigma^{\frown}{}_{\kappa}{}^{\dot G H}\cdots.\end{aligned} \tag{2.9}$$

§ 3. SL(2, C) TRANSFORMATIONS

Before going on, we need to make clear what the statement, "the spacetime manifold M is flat" means in our case. We shall say so whenever we can find the *global**) coordinate $\{x^{\mu}\}$ (called the Minkowski coordinate here) in terms of which the coordinate basis ($\partial_{\underline{\mu}} \equiv \frac{\partial}{\partial x^{\mu}}$) is coincided with a Lorentz basis all over M.*)

Let's start on a discussion by assuming that the spacetime M is, at the outset, flat. We shall therefore be able to find Minkowski coordinate x^{μ} such as

$\mathbf{E}_{\underline{\mu}} = \partial_{\underline{\mu}}$ throughout all points of M.

Owing to the relation (2. 1) the Lorentz basis $\{\mathbf{E}_{\underline{\mu}}\}$ is transformed under the SL(2, C) spin-basis transformations

$$\omega'_{A} = \omega_{B}(S^{-1})^{B}{}_{A} \quad \text{or} \quad \omega'^{A} = S^{A}{}_{B}\omega^{B} \tag{3. 1}$$

like

$$\mathbf{E}'_{\underline{\mu}} = \mathbf{E}_{\underline{\nu}}(L^{-1})^{\nu}{}_{\mu} \quad \text{or} \quad \mathbf{E}'^{\underline{\mu}} = L^{\mu}{}_{\nu}\mathbf{E}^{\underline{\nu}}, \tag{3. 2}$$

where $L^{\mu}{}_{\nu}$ is related to $S^{A}{}_{B}$ by

$$L^{\mu}{}_{\nu} = \sigma^{\underline{\mu}}{}_{\dot{A}B} S^{\dot{A}}{}_{\dot{C}} S^{B}{}_{D} \sigma_{\underline{\nu}}{}^{\dot{C}D}. \tag{3. 3}$$

Of course, this is one of Lorentz transformations.

Then we shall also be able to find the new Minkowski coordinate $\{x'^{\mu}\}$ such as

$$\mathbf{E}'_{\underline{\mu}} = \partial'_{\underline{\mu}} \tag{3. 4}$$

throughout all points of M.

Here the new coordinate $\{x'^{\mu}\}$ is given in terms of the old $\{x^{\mu}\}$ by

$$x'^{\underline{\mu}} = L^{\mu}{}_{\nu}x^{\underline{\nu}}. \tag{3. 5}$$

Generally speaking, as long as we treat with the global**) SL(2, C) spin-basis transformations we shall always be able to find the corresponding Minkowski coordinate in a flat spacetime.

On the contrary, when we treat with the "local" SL(2, C) spin-basis transformations, in which $S^{A}{}_{B}$ depends explicitly upon the coordinates, then we shall no longer be able to find the corresponding Minkowski coordinate. In this case, the Lorentz basis is transformed like

$$\mathbf{E}'_{\underline{\mu}} = \mathbf{E}_{\underline{\nu}}(L^{-1}(x))^{\nu}{}_{\mu}, \tag{3. 6}$$

where $L^{\nu}{}_{\mu}(x)$ is related to $S^{A}{}_{B}(x)$ in the same way as (3. 3):

$$L^{\mu}{}_{\nu}(x) = \sigma^{\mu}{}_{\dot{A}C} S^{\dot{A}}{}_{\dot{B}}(x) S^{C}{}_{D}(x) \sigma_{\underline{\nu}}{}^{\dot{B}D}. \tag{3. 7}$$

Consequently, the condition for a Lorentz basis

$$\mathbf{g}(\mathbf{E}_{\underline{\mu}}, \mathbf{E}_{\underline{\nu}}) = \mathbf{g}(\mathbf{E}'_{\underline{\mu}}, \mathbf{E}'_{\underline{\nu}}) = \eta_{\mu\nu}$$

is also valid as before.

However, we can no longer find the coordinate basis to be coincided with this

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* It is always possible to find the coordinate basis coincided *locally* with a Lorentz basis.

** The terms "global" means $S^{A}{}_{B} = \text{const}$.

Lorentz basis all over M.

According to our assumption, we conclude that the spacetime M is no longer flat in this case.

We shall then have the following question.

"What does the Minkowski coordinate x^{μ} with which we started on our discussion become now?"

Perhaps, we shall be able to answer this question in various ways. Nevertheless, it seems to be most natural to consider that the Minkowski coordinate x^{μ} is transformed like

$$x'^{\mu}=L^{\mu}{}_{\nu}(x)x^{\nu}\,, \tag{3. 8}$$

corresponding to the transformation of a Lorentz basis. Therefore, we shall then consider that a coordinate basis $\{\partial_{\underline{\mu}}\}$ is transformed as follows;

$$\partial_{\underline{\mu}}=\{L^{\nu}{}_{\kappa}(x),{}_{\mu}x^{\kappa}+L^{\nu}{}_{\mu}(x)\}\partial'_{\nu}\,. \tag{3. 9}$$

Lastly, in order to use later we shall here give the infinitesimal forms of the various equations obtained above.

First of all, let $S^{A}{}_{B}\simeq\delta^{A}{}_{B}+\eta^{A}{}_{B}\,,$ (3. 10)

where $\eta_{AB}(\equiv\eta^{C}{}_{B}\epsilon_{CA})$ is an infinitesimal parameter of SL(2, C) transformations and symmetric in the indices A and B.

We then find

$$L^{\mu}{}_{\nu}\simeq\delta^{\mu}{}_{\nu}+\omega^{\mu}{}_{\nu}\,,$$

where

$$\omega^{\mu}{}_{\nu}\equiv\eta^{B}{}_{D}\sigma^{\mu}{}_{\dot{A}B}\sigma_{\underline{\nu}}{}^{\dot{A}D}+\eta^{\dot{A}}{}_{\dot{C}}\sigma^{\mu}{}_{\dot{A}B}\sigma_{\underline{\nu}}{}^{\dot{C}B}\,, \tag{3. 12}$$

which is an infinitesimal skew-symmetric parameter of the Lorentz transformations.

Now, a coordinate basis $\partial_{\underline{\mu}}$ is infinitesimally transformed like

$$\partial'_{\mu}=\partial_{\underline{\mu}}-\xi^{\lambda},{}_{\mu}\partial_{\underline{\lambda}}\,, \tag{3. 13}$$

corresponding to the Lorentz basis $\{\mathbf{E}_{\underline{\mu}}\}$ which is transformed like

$$\mathbf{E}'_{\underline{\mu}}=\mathbf{E}_{\underline{\mu}}-\omega^{\lambda}{}_{\mu}\mathbf{E}_{\underline{\lambda}}\,, \tag{3. 14}$$

where ξ^{λ} is the variation of coordinates, that is,

$$\xi^{\lambda}\equiv\delta x^{\lambda}=\omega^{\lambda}{}_{\kappa}(x)x^{\kappa}\,. \tag{3. 15}$$

§ 4. THE ACTION PRINCIPLE

We consider a set of field variables $q_{A}(x)$, which we regard as the elements of a column matrix $q(x)$, with the Lagrangian

$$L(q,\ q,_{\underline{\mu}},\ x), \tag{4. 1}$$

where x^μ is, of course, a Minkowski coordinate.*)

The action integral over an arbitrary spacetime region Σ

$$I(\Sigma)=\int_\Sigma L(x)d^4x \tag{4. 2}$$

is invariant under the following transformation

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu + \delta x^\mu, \\ q &\to q'(x') = q(x) + \delta q(x) \end{aligned} \tag{4. 3}$$

if

$$\delta L + L(\delta x^\mu),_\mu = \delta^* L + (L\delta x^\mu),_\mu = 0 \tag{4. 4}$$

at any spacetime points, where δ^* means a substantial variation i.e.,

$$\delta^* L \equiv L'(x) - L(x) = \delta L - L,_\mu \delta x^\mu. \tag{4. 5}$$

Now, let us postulate that the action integral (4. 2) is invariant under the inhomogeneous Lorentz transformation; **)

$$\begin{aligned} \delta x^\mu &= \omega^\mu{}_\nu x^\nu + \varepsilon^\mu, \\ \delta q &= \tfrac{1}{2}\omega^{\mu\nu} S_{\mu\nu} q. \end{aligned} \tag{4. 6}$$

Here $\omega^{\mu\nu}$ and ε^μ are infinitesimal parameters of Poincaré group, and $S_{\mu\nu}$ is the representation matrix satisfying commutation rules appropriate to the generators of the homogeneous Lorentz group, i.e.,

$$\begin{aligned} &S_{\mu\nu} = -S_{\nu\mu}, \\ &[S_{\mu\nu}, S_{\rho\sigma}] = \tfrac{1}{2} f_{\mu\nu}{}^{\lambda\kappa}{}_{\rho\sigma} S_{\lambda\kappa}, \\ &f_{\mu\nu}{}^{\lambda\kappa}{}_{\rho\sigma} = (\delta^\lambda_\mu \delta^\kappa_\sigma - \delta^\lambda_\sigma \delta^\kappa_\mu)\eta_{\nu\rho} + (\delta^\lambda_\nu \delta^\kappa_\rho - \delta^\lambda_\rho \delta^\kappa_\nu)\eta_{\mu\sigma} + (\delta^\lambda_\sigma \delta^\kappa_\nu - \delta^\lambda_\nu \delta^\kappa_\sigma)\eta_{\mu\rho} + (\delta^\lambda_\rho \delta^\kappa_\mu - \delta^\lambda_\mu \delta^\kappa_\rho)\eta_{\nu\sigma}. \end{aligned} \tag{4. 7}$$

In this case the condition (4. 4) reduces to $\delta L = 0$, since $(\delta x^\mu),_\mu = \omega^\mu{}_\mu = 0$, and yields the following 10 identities

$$\frac{\partial L}{\partial x^\mu} \equiv 0, \tag{4. 8}$$

$$\frac{\partial L}{\partial q} S_{\mu\nu} q + \frac{\partial L}{\partial q,_\lambda}(S_{\mu\nu} q,_\lambda + \eta_{\lambda\mu} q,_\nu - \eta_{\lambda\nu} q,_\mu) \equiv 0.^{***)} \tag{4. 9}$$

(4. 8) stands for the requirement that L does not explicitly depend on x, as might be expected from translational invariance.

Alternatively, we shall obtain the following equations to start with the 2nd of eq. (4. 4)

$$[L]\delta^* q - \tfrac{1}{2} M^\mu{}_{\lambda\kappa},_\mu \omega^{\lambda\kappa} - T^\mu{}_\nu,_\mu \varepsilon^\nu = 0, \tag{4. 10}$$

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* A Lorentz basis (or a spin basis) is not given explicitly in this section. But it should be understood that a set of fields q is the set of their components with respect to a Lorentz basis (or a spin basis).

** Of course, the Minkowskian property of the coordinate is also preserved under the inhomogeneous Lorentz transformation.

*** Note that $\partial/\partial q$ must be regarded as a row matrix.

where

$$[L] \equiv \frac{\partial L}{\partial q} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial q,_{\mu}}\right),$$

$$M^\mu{}_{\nu\rho} \equiv S^\mu{}_{\nu\rho} + T^\mu{}_\nu x_\rho - T^\mu{}_\rho x_\nu,$$

$$S^\mu{}_{\nu\rho} \equiv -\frac{\partial L}{\partial q,_{\mu}} S_{\nu\rho} q,$$

$$T^\mu{}_\nu \equiv \frac{\partial L}{\partial q,_{\mu}} q,_{\nu} - L\delta^\mu{}_\nu. \tag{4. 11}$$

Then taking into account the field equation $[L]=0$, we obtain the following conservation laws

$$M^\mu{}_{\nu\rho},_{\mu}=0, \qquad T^\mu{}_\nu,_{\mu}=0. \tag{4. 12}$$

Next, let us consider the local SL(2, C) transformations from the viewpoint discussed in the previous section.

In this case q and $q,_{\mu}$ are transformed like

$$\delta q = \tfrac{1}{2}\omega^{\alpha\beta}(x) S_{\alpha\beta} q \tag{4. 13}$$

and

$$\begin{aligned}\delta q,_{\mu} &= (\delta q),_{\mu} - \xi^\lambda,_{\mu} q,_{\lambda} \\ &= \tfrac{1}{2}\omega^{\lambda\kappa}(x) S_{\lambda\kappa} q,_{\mu} + \tfrac{1}{2}\omega^{\lambda\kappa},_{\mu}(x) S_{\lambda\kappa} q - \xi^\lambda,_{\mu} q,_{\lambda},\end{aligned} \tag{4. 14}$$

respectively and then, we see that the condition (4. 4) is no longer satisfied by the original Lagrangian (4. 1):

$$\begin{aligned}\delta L + L(\delta x^\mu),_{\mu} &= \delta^* L + (L\delta x^\mu),_{\mu} \\ &= [L]\delta^* q - \tfrac{1}{2}\omega^{\alpha\beta} M^\mu{}_{\alpha\beta},_{\mu} - \tfrac{1}{2}\omega^{\alpha\beta},_{\mu} M^\mu{}_{\alpha\beta} \\ &= -\tfrac{1}{2}\omega^{\alpha\beta},_{\mu} M^\mu{}_{\alpha\beta} \neq 0.\end{aligned}$$

Therefore we must look for a modified Lagrangian which makes the action integral invariant.

It is natural to seek a invariant action in those form that

$$I'(\Sigma) = \int_\Sigma L' D d^4x, \tag{4. 15}$$

because d^4x is not a scalar under the transformations considered here. The condition (4. 4) becomes now

$$\delta L' = 0, \qquad \delta D + D\xi^\mu,_{\mu} = 0 \tag{4. 16}$$

since Dd^4x is a scalar.

Furthermore, we take note of the following.

We saw in the previous section that a nonflat spacetime is induced by the local SL(2, C) transformations and we obtained there a coordinate basis which has the transformation rule (3. 13) besides a Lorentz basis. Now we shall be allowed for us to introduce a field $h_{\underline{\mu}}{}^{\nu}$ (or $h^{\underline{\nu}}{}_{\mu}$) here. Because we can find the field $h_{\underline{\mu}}{}^{\nu}$ as the intermediate owing to (2. 5) whenever we have both a coordinate and a Lorentz basis.

The transformation properties of $h_{\underline{\mu}}{}^{\nu}$ (or $h^{\underline{\nu}}{}_{\mu}$) are obtained by making use of (3. 13), (3. 14) and (2. 5);

$$\delta h_{\underline{\mu}}{}^{\nu} = -\omega^{\lambda}{}_{\mu} h_{\underline{\lambda}}{}^{\nu} + \xi^{\nu},{}_{\lambda} h_{\underline{\mu}}{}^{\lambda}$$

or

$$\delta h^{\underline{\nu}}{}_{\mu} = \omega^{\nu}{}_{\lambda} h^{\underline{\lambda}}{}_{\mu} - \xi^{\lambda},{}_{\mu} h^{\underline{\nu}}{}_{\lambda}\,. \tag{4. 17}$$

Taking into account this note, we could assume the invariant Lagrangian L′ as follows:

$$L' = L'(q,\ q,{}_{\mu},\ h_{\mu|}{}^{\nu},\ h_{\mu|}{}^{\nu},{}_{\lambda})^{*),\ **)} \tag{4. 18}$$

Then, from the invariance postulate we get the following identity:

$$\delta L'(x) \equiv \frac{\partial L'}{\partial q}\delta q + \frac{\partial L'}{\partial q,{}_{\mu}}\delta q,{}_{\mu} + \frac{\partial L'}{\partial h_{\mu|}{}^{\nu}}\delta h_{\mu|}{}^{\nu} + \frac{\partial L'}{\partial h_{\mu|}{}^{\nu},{}_{\lambda}}\delta h_{\mu|}{}^{\nu},{}_{\lambda} \equiv 0.$$

Inserting (4. 13), (4. 14) and (4. 17) into the above, and taking account of the arbitrariness of choosing $\omega^{\mu\nu}$, $\omega^{\mu\nu},{}_{\lambda}$, $\omega^{\mu\nu},{}_{\lambda\kappa}$ and the fact that it must be valid at any points of M, we are led to the following various identities

$$\frac{\partial L'}{\partial q,{}_{\mu}} S_{\alpha\beta} q + \frac{\partial L'}{\partial h^{\alpha|\nu},{}_{\mu}} h_{\beta|}{}^{\nu} - \frac{\partial L'}{\partial h^{\beta|\nu},{}_{\mu}} h_{\alpha|}{}^{\nu} \equiv 0, \tag{4. 19}$$

$$\frac{\partial L'}{\partial q} S_{\alpha\beta} q + \frac{\partial L'}{\partial q,{}_{\mu}} S_{\alpha\beta} q,{}_{\mu} + \frac{\partial L'}{\partial h^{\alpha|\nu}} h_{\beta|}{}^{\nu} - \frac{\partial L'}{\partial h^{\beta|\nu}} h_{\alpha|}{}^{\nu} + \frac{\partial L'}{\partial h^{\alpha|\nu},{}_{\lambda}} h_{\beta|}{}^{\nu},{}_{\lambda} - \frac{\partial L'}{\partial h^{\beta|\nu},{}_{\lambda}} h_{\alpha|}{}^{\nu},{}_{\lambda} \equiv 0, \tag{4. 20}$$

$$\frac{\partial L'}{\partial q,{}_{\mu}} q,{}_{\alpha} - \frac{\partial L'}{\partial h^{\gamma|\alpha}} h^{\gamma|\mu} - \frac{\partial L'}{\partial h^{\gamma|\alpha},{}_{\lambda}} h^{\gamma|\mu},{}_{\lambda} + \frac{\partial L'}{\partial h^{\gamma|\nu},{}_{\mu}} h^{\gamma|\nu},{}_{\alpha} \equiv 0, \tag{4. 21}$$

$$\frac{\partial L'}{\partial h^{\gamma|\alpha},{}_{\sigma}} h^{\gamma|\mu} + \frac{\partial L'}{\partial h^{\gamma|\alpha},{}_{\mu}} h^{\gamma|\sigma} \equiv 0, \tag{4. 22}$$

where $h^{\alpha|\mu} \equiv \eta^{\alpha\beta} h_{\beta|}{}^{\mu}$.

Here we should stress that these results agree with those ones that are treated as if $\omega^{\lambda\kappa}$ and ξ^{μ} are independent parameters. Consequently, our final results will be also invariant under the general coordinate transformations.

Now, we see at once that from (4. 22), $h_{\mu|}{}^{\nu},{}_{\lambda}$ should be contained in L′ only through the combination

$$C^{\xi\eta\zeta} \equiv h^{\alpha|\beta},{}_{\gamma} h^{\xi|}{}_{\beta} (h^{\eta|\gamma}\delta^{\zeta}_{\alpha} - \delta^{\eta}_{\alpha} h^{\zeta|\gamma}), \tag{4. 23}$$

where we should note the relations***)

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* From the invariance requirements L′ needs to depend upon $h,{}_{\rho}$ as well as h,

** Here and in the following, the notations $h_{\mu|}{}^{\nu}$, $h^{\nu|}{}_{\mu}$ are used for $h_{\underline{\mu}}{}^{\nu}$, $h^{\underline{\nu}}{}_{\mu}$ respectively.

*** Because $h_{\mu|\nu}$ is a inverse of $h^{\mu|\nu}$.

$$h^{\alpha|\lambda}h_{\alpha|\kappa} = \delta^{\lambda}_{\kappa}, \qquad h_{\alpha|\lambda}h^{\beta|\lambda}=\delta^{\beta}_{\alpha}. \tag{4. 24}$$

Therefore we put

$$L'(q,\ q,_{\mu},\ h,\ h,_{\mu})=L''(q,\ q,_{\mu},\ h,\ C).$$

Then (4. 19), (4. 20) and (4. 21) are rewritten respectively as follows;

$$\frac{1}{2}\frac{\partial L''}{\partial q,_{\mu}}h_{\eta|\mu}S_{\alpha\beta}q+\frac{\partial L''}{\partial C^{\beta\eta\alpha}}-\frac{\partial L''}{\partial C^{\alpha\eta\beta}}=0, \tag{4. 25}$$

$$\frac{1}{2}\left(\frac{\partial L''}{\partial q}S_{\alpha\beta}q+\frac{\partial L''}{\partial q,_{\mu}}S_{\alpha\beta}q,_{\mu}\right)+\frac{\partial L''}{\partial h^{\alpha|\nu}}h_{\beta}{}^{\nu}+\frac{\partial L''}{\partial C^{\alpha\xi\gamma}}C_{\beta}{}^{\xi\gamma}+2\frac{\partial L''}{\partial C^{\xi\alpha\gamma}}C^{\xi}{}_{\beta}{}^{\gamma}$$
$$-\{\alpha \text{ and } \beta \text{ interchanged}\}\equiv 0 \tag{4. 26}$$

and

$$\frac{\partial L''}{\partial q,_{\mu}}q,_{\alpha}-\frac{\partial L''}{\partial h^{\gamma|\alpha}}h^{\gamma|\mu}\equiv 0. \tag{4. 27}$$

From (4. 27) we see at once that $h_{\mu|}{}^{\nu}$ and $q,_{\lambda}$ should be contained in L″ through the combination

$$q_{|\varepsilon}\equiv h_{\varepsilon|}{}^{\beta}q,_{\beta}. \tag{4. 28}$$

Therefore, we put furthermore

$$L''(q,\ q,_{\mu},\ h,\ C)\equiv L'''(q,\ q_{|\mu},\ C).$$

Then (4. 25) and (4. 26) are rewritten again as

$$\frac{1}{2}\frac{\partial L'''}{\partial q_{|\varepsilon}}\eta_{\varepsilon\mu}S_{\alpha\beta}q+\frac{\partial L'''}{\partial C^{\beta\eta\alpha}}-\frac{\partial L'''}{\partial C^{\alpha\eta\beta}}\equiv 0 \tag{4. 29}$$

and

$$\frac{1}{2}\left\{\frac{\partial L'''}{\partial q}S_{\alpha\beta}q+\frac{\partial L'''}{\partial q_{|\varepsilon}}(S_{\alpha\beta}q_{|\varepsilon}+\eta_{\alpha\varepsilon}q_{|\beta}-\eta_{\beta\varepsilon}q_{|\alpha})\right\}$$
$$+\frac{\partial L'''}{\partial C^{\alpha\xi\gamma}}C_{\beta}{}^{\xi\gamma}+2\frac{\partial L'''}{\partial C^{\xi\alpha\gamma}}C^{\xi}{}_{\beta}{}^{\gamma}-\{\alpha \text{ and } \beta \text{ interchanged}\}\equiv 0. \tag{4. 30}$$

(4. 29) shows that L‴ should depend upon $q_{|\varepsilon}$ and $C^{\xi\eta\zeta}$ through the combination

$$q_{;\varepsilon}\equiv q_{|\varepsilon}+\tfrac{1}{2}A^{\lambda\kappa}{}_{\varepsilon}S_{\lambda\kappa}q,$$

where

$$A^{\lambda\kappa}{}_{\varepsilon}\equiv\tfrac{1}{2}(C^{\xi\lambda\kappa}+C^{\kappa\lambda\xi}-C^{\lambda\kappa\xi})\eta_{\xi\varepsilon}$$
$$=\tfrac{1}{2}(C_{\varepsilon}{}^{\lambda\kappa}+C^{\kappa\lambda}{}_{\varepsilon}-C^{\lambda\kappa}{}_{\varepsilon}) \tag{4. 32}$$

Therefore, if we put*)

$$L'''(q,\ q_{|\mu},\ C)\equiv L(q,\ q_{;\mu}),$$

We shall find again (4. 9) with $q_{;\mu}$ in place of $q,_{\mu}$:

$$\frac{\partial L}{\partial q}S_{\mu\nu}q+\frac{\partial L}{\partial q_{;\lambda}}(S_{\mu\nu}q_{;\lambda}+\eta_{\lambda\mu}q_{;\nu}-\eta_{\lambda\nu}q_{;\mu})\equiv 0$$

In fact, (4. 30) is just rewritten into this form by noting the relations

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* This choice is due to the requirement that when the field $h_{\alpha|}{}^{\beta}$ is assumed to be δ^{β}_{α}, we must have the original action integral. This requirement also, at the same time, implies that *D* is unity in the case.

$$\frac{\partial L'''}{\partial q}=\frac{\partial L}{\partial q}+\frac{\partial L}{\partial q_{;\mu}}\tfrac{1}{2}A^{\lambda\kappa}{}_{\mu}S_{\lambda\kappa},$$

$$\frac{\partial L'''}{\partial C^{\xi\eta\gamma}}=\frac{\partial L}{\partial q_{;\mu}}\tfrac{1}{4}(\eta_{\xi\mu}S_{\eta\gamma}+\eta_{\gamma\mu}S_{\eta\xi}-\eta_{\eta\mu}S_{\gamma\xi})q,$$

$$\frac{\partial L'''}{\partial q_{|\mu}}=\frac{\partial L}{\partial q_{;\mu}}$$

Thus we have obtained the required Lagrangian but not yet the invariant action.

In order to obtain the invariant action it is sufficient to notice that $\sqrt{-g}$ is transformed like

$$\delta\sqrt{-g}=-\sqrt{-g}\,\xi^{\mu}{}_{,\mu}, \tag{4. 33}$$

where $g=\det(g_{\mu\nu})=\det(\eta^{\alpha\beta}h_{\alpha|\mu}h_{\beta|\nu})$. Consequently, we can choose $\sqrt{-g}$ for D.

Next, let us investigate the possible type of the invariant Lagrangian for the free h-field.

First of all, we must assume the invariant form as follows:

$L_0(h_{\mu|}{}^{\nu},\ h_{\mu|}{}^{\nu}{}_{,\lambda},\ h_{\mu|}{}^{\nu}{}_{,\lambda\kappa})$*).

From the invariance postulate we obtain the following identities**)

$$\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu\alpha}}h^{\gamma|\lambda}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\lambda\alpha}}h^{\gamma|\nu}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu\lambda}}h^{\gamma|\alpha}\equiv 0, \tag{4. 34}$$

$$\frac{\partial L_0}{\partial h^{\mu|\gamma}{}_{,\nu\lambda}}h_{\alpha|}{}^{\gamma}-\frac{\partial L_0}{\partial h^{\alpha|\gamma}{}_{,\nu\lambda}}h_{\mu|}{}^{\gamma}\equiv 0, \tag{4. 35}$$

$$\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu}}h^{\gamma|\lambda}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\lambda}}h^{\gamma|\nu}+2\left(\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu\alpha}}h^{\gamma|\lambda}{}_{,\alpha}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\lambda\alpha}}h^{\gamma|\nu}{}_{,\alpha}-\frac{\partial L_0}{\partial h^{\gamma|\sigma}{}_{,\nu\lambda}}h^{\gamma|\sigma}{}_{,\mu}\right)\equiv 0, \tag{4. 36}$$

$$\frac{\partial L_0}{\partial h^{\gamma|\mu}}h^{\gamma|\lambda}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu}}h^{\gamma|\lambda}{}_{,\nu}-\frac{\partial L_0}{\partial h^{\gamma|\nu}{}_{,\lambda}}h^{\gamma|\nu}{}_{,\mu}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu\alpha}}h^{\gamma|\lambda}{}_{,\nu\alpha}-2\frac{\partial L_0}{\partial h^{\gamma|\nu}{}_{,\lambda\alpha}}h^{\gamma|\nu}{}_{,\mu\alpha}\equiv 0, \tag{4. 37}$$

$$\frac{\partial L_0}{\partial h^{\mu|\gamma}{}_{,\lambda}}h_{\beta|}{}^{\gamma}-\frac{\partial L_0}{\partial h^{\beta|\gamma}{}_{,\lambda}}h_{\mu|}{}^{\gamma}+2\left(\frac{\partial L_0}{\partial h^{\mu|\gamma}{}_{,\lambda\alpha}}h_{\beta|}{}^{\gamma}{}_{,\alpha}-\frac{\partial L_0}{\partial h^{\beta|\gamma}{}_{,\lambda\alpha}}h_{\mu|}{}^{\gamma}{}_{,\alpha}\right)\equiv 0 \tag{4. 38}$$

and

$$\frac{\partial L_0}{\partial h^{\gamma|\mu}}h_{\beta|}{}^{\mu}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu}}h_{\beta|}{}^{\mu}{}_{,\nu}+\frac{\partial L_0}{\partial h^{\gamma|\mu}{}_{,\nu\alpha}}h_{\beta|}{}^{\mu}{}_{,\nu\alpha}-\{\gamma\text{ and }\beta\text{ interchanged}\}\equiv 0. \tag{4. 39}$$

After the tedious calculations, from (4. 34), (4. 35) and (4. 38) we see that $h_{\mu|}{}^{\nu}{}_{,\lambda}$ should be contained in L_0 through the combination***)

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* The invariant Lagrangian which is independent on the 2nd derivatives of $h_{\mu|}{}^{\nu}$ does not exist. This can be seen to note that (4. 38) and (4. 39) together with $\partial L_0/\partial h_{\mu|}{}^{\nu}{}_{,\lambda\kappa}=0$ are not satisfied at the same time.

** The note to follow (4. 22) holds also for these identities.

*** Note $A^{\xi\eta|}{}_{\varepsilon}\equiv A^{\xi\eta}{}_{\zeta}h^{\zeta|}{}_{\varepsilon}$.

$$F^{\xi\eta|}{}_{\varepsilon\omega} \equiv A^{\xi\eta|}{}_{\varepsilon,\,\omega} - A^{\xi\eta|}{}_{\omega,\,\varepsilon} - A^{\xi}{}_{\delta|\varepsilon} A^{\delta\eta|}{}_{\omega} + A^{\xi}{}_{\delta|\omega} A^{\delta\eta|}{}_{\varepsilon}. \tag{4. 40}$$

Therefore, let's put

$$L_0(h,\ h,_{\mu},\ h,_{\mu\nu}) = L_0'(h,\ F)$$

and investigate the remainders (4. 37) and (4. 39).

From (4. 37) it is easy to see that $h_{\mu|}{}^{\nu}$ and $F^{\xi\eta|}{}_{\varepsilon\omega}$ must be contained in L_0' through the combination

$$R^{\xi\eta}{}_{\alpha\beta} \equiv F^{\xi\eta|}{}_{\varepsilon\omega} h_{\alpha|}{}^{\varepsilon} h_{\beta|}{}^{\omega}. \tag{4. 41}$$

Then (4. 39) can be rewritten in terms of R as

$$\frac{\partial L_0}{\partial R^{\xi\eta}{}_{\lambda\kappa}} \{\delta^{\xi}_{\gamma} R_{\beta}{}^{\eta}{}_{\lambda\kappa} + \delta^{\eta}_{\gamma} R^{\xi}{}_{\beta\lambda\kappa} + \eta_{\lambda\gamma} R^{\xi\eta}{}_{\beta\kappa} + \eta_{\kappa\gamma} R^{\xi\eta}{}_{\lambda\beta} - (\beta \text{ and } \gamma \text{ interchanged})\} \equiv 0. \tag{4. 42}$$

In order to look for the explicit R-dependence of L_0, we attend to the transformation property of R. This is given by

$$\delta R^{\xi\eta}{}_{\alpha\beta} = \omega^{\xi}{}_{\gamma} R^{\gamma\eta}{}_{\alpha\beta} + \omega^{\eta}{}_{\gamma} R^{\xi\gamma}{}_{\alpha\beta} - \omega^{\gamma}{}_{\alpha} R^{\xi\gamma}{}_{\alpha\beta} - \omega^{\gamma}{}_{\beta} R^{\xi\eta}{}_{\alpha\gamma}, \tag{4. 43}$$

since $A^{\lambda\kappa|}{}_{\mu}$ is transformed like

$$\delta A^{\lambda\kappa|}{}_{\mu} = \omega^{\lambda}{}_{\alpha} A^{\alpha\kappa|}{}_{\mu} + \omega^{\kappa}{}_{\alpha} A^{\lambda\sigma|}{}_{\mu} - \xi^{\nu},{}_{\mu} A^{\lambda\kappa|}{}_{\nu} - \omega^{\lambda\kappa},{}_{\mu}. \tag{4. 44}$$

Then, comparing (4. 43) with the expression $\{\cdots\}$ on the left-hand side of (4. 42) we shall find at once that (4. 42) is written like

$$\frac{\partial L_0}{\partial R^{\xi\mu}{}_{\lambda\kappa}} \delta R^{\xi\eta}{}_{\lambda\kappa} \equiv 0.$$

This shows that L_0 can be the arbitrary invariant function of $R^{\xi\eta}{}_{\lambda\kappa}$.

§ 5. CONCLUSION

We have seen in the preceding sections that the invariant action under the local SL(2, C) transformations is written in perfect form as

$$I(\Sigma) = \int_{\Sigma} L_T d^4x, \tag{5. 1}$$

where

$$L_T \equiv \sqrt{-g}\{L(q,\ q_{;\mu}) + L_0(R)\},$$

$$q_{;\mu} \equiv h_{\mu|}{}^{\nu} \nabla_{\nu} q,$$

$$\nabla_{\nu} q \equiv q,_{\nu} + \tfrac{1}{2} A^{\lambda\kappa|}{}_{\nu} S_{\lambda\kappa} q,$$

$$A^{\lambda\kappa|}{}_{\mu} = \left\{ {\alpha \atop \beta\mu} \right\} h^{\lambda|}{}_{\alpha} h^{\kappa|\beta} + h^{\lambda|}{}_{\alpha} h^{\kappa|\alpha},{}_{\mu},$$

$$\left\{ {\alpha \atop \beta\mu} \right\} = \text{the Christoffel symbol of the 2nd kind},$$

and $L_0(R)$ is an arbitrary invariant function to be made out of $R^{\xi\eta}{}_{\lambda\kappa}$.

Now, let us make the suitable choice for the matter field Lagrangian L and choose a particular Lagrangian*) for L_0

$$L_0 = R = R^{\xi\eta}{}_{\xi\eta}\,.$$

Then taking a variation of (5. 1) with respect to the field $h_{\mu|}{}^{\nu}$ we shall find the Einstein's gravitational field equations extended so as to be able to include a spinor.[1] That it is so is obtained from the fact that R is just equal to the Riemann scalar, because

$$R = R^{\xi\eta}{}_{\xi\eta} = R^{\alpha\beta}{}_{\alpha\beta}\,,$$

where

$$R^{\alpha}{}_{\beta\varepsilon\omega} = F^{\xi\eta|}{}_{\varepsilon\omega} h_{\xi|}{}^{\alpha} h_{\eta|\beta} = R^{\xi\eta}{}_{\gamma\delta} h_{\xi|}{}^{\alpha} h_{\eta|\beta} h^{\gamma|}{}_{\varepsilon} h^{\delta|}{}_{\omega}$$

$$= \left\{ {\alpha \atop \beta\varepsilon} \right\}_{,\,\omega} - \left\{ {\alpha \atop \beta\omega} \right\}_{,\,\varepsilon} - \left\{ {\alpha \atop \chi\varepsilon} \right\}\left\{ {\chi \atop \beta\omega} \right\} + \left\{ {\alpha \atop \chi\omega} \right\}\left\{ {\chi \atop \beta\varepsilon} \right\}$$

wihch is just the Riemann curvature tensor.

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* In more general we can also choose the Lagrangian L_0 so as to contain the quadratic terms in $R^{\xi\eta}{}_{\varepsilon\omega}$.[9]

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