THE THEORY OF FUNCTIONS OF AN OCTONION VARIABLE

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Abstract

There has been an increasing interest among some physicists in the mathematical theory of octonions, especially in their algebraic characters which are closely related to the quark model. Octonions obey neither the commutative nor the associative laws and was thought that the theory of functions of an octonion variable may differ greatly from that of a complex variable and that of a quaternion variable. However, Dentoni and Sce have shown that the theorems and the methods in the theory of functions of a quaternion variable can be extended to that of an octonion variable. In this paper we describe the results obtained by Dentoni and Sce in a more general form, and develop the theory of regular polynomial functions of an octonion variable which are an extension of Fueter's polynomials in the quaternion theory.

1. Introduction.

The discovery of octonions was made by Cayley[1] and independently by Graves[2], immediately after Hamilton's discovery of quaternions[3]. The algebraic theory of octonions, which is called Cayley algebra, assumes neither the commutative law nor the associative law, i. e.,

$$AB \neq BA \quad \text{and} \quad (AB)C \neq A(BC),$$

where A, B, C are octonions. The theory of functions of a quaternion variable was developed by Fueter[4], but it had been thought that the theory of functions of an octonion variable would be quite different from that of a quaternion variable, because of the violation of the associative law in octonions. There has been an increasing interest among some physicists in the theory of octonions; especially the algebra of octonions which comprises the quark structure; see Günaydin, Gürsey, Tze, Okubo and Morita[5]. In the same year Dentoni and Sce[6] developed the theory of functions of an octonion variable in a similar manner to that of a quaternion variable. Since the theory of functions of a complex-quaternion variable comprises the theory of electromagnetic fields[7] we might expect that the theory of functions of an octonion variable comprises not only the algebraic aspects but also the dynamical aspects of the quark theory. For this purpose we need to extend the theory of functions of an octonion variable to that of a complex octonions (split octonions), just as in the case of the emergence of Maxwell's theory from the theory of functions of a complex quaternion variable.

In this paper, as a first step in the development of the theory, we briefly describe the theory of Dentoni and Sce in a more general form and introduce the regular polynomial functions of an octonion variable in a similar manner to tbat of a biquaternion variable[7].

The contents of the paper are as follows: —
In section 2, we briefly look at the algebra of octonions and introduce the notation. In section 3, we discuss the regularity condition for a function of an octonion variable. In section 4, we look at functional derivatives and study the regularity conditions in a functional form. In section 5, we construct a regular function. In section 6, we derive the theorem of residues and in section 7 we introduce regular polynomial functions of an octonion variable.

2. Algebra of octonions (Cayley algebra).

An eight-dimensional Cayley-Graves algebra of octonions, $\mathbf{O}$, is a direct sum of two copies of quaternions, $\mathbf{Q}$[8], i.e.,

(1) $\quad \mathbf{O}=\mathbf{Q}\oplus\mathbf{Q}$

An octonion c is defined by a pair of quaternions, Q_1 and Q_2 as

(2) $\quad c=(Q_1, Q_2), \quad Q_1, Q_2\in\mathbf{Q}.$

The addition and the multiplication of two octonions, $c=(Q_1, Q_2)$ and $c'=(Q_1', Q'_2)$, are defined as follows:

(3) $\quad c+c'=(Q_1, Q_2)+(Q_1', Q_2')=(Q_1+Q_1', Q_2'+Q_2'),$

(4) $\quad cc'=(Q_1, Q_2)(Q_1', Q_2')=(Q_1Q_1'-\overline{Q}_2'Q_2, Q_2\overline{Q}_1'+Q_2'Q_1),$

where $\overline{Q}'_1$ and $\overline{Q}_2'$ are the conjugates of Q_1' and Q_2', respectively[*]. In equation (4) one should keep in mind that all the Q's are quaternions so that their products are noncommutative, i.e., $Q_1Q_1'\neq Q_1'Q_1$, and so on. Using the definitions (2) and

(4) we now define the basis of $\mathbf{O}$ by

$$(8)\qquad 1=(1,\ 0),\ i_1=(i_1,\ 0),\ i_2=(i_2,\ 0),\ i_{3},=(i_3,\ 0),$$
$$i_4=(0,\ 1),\ i_5=(0,\ i_1),\ i_6=(0,\ i_2),\ i_7=(0,\ i_3).$$

(*) A quaternion Q is defined as

$$(5)\qquad Q=q_0+q_1i_1+q_2i_2+q_3i_3,$$

where 1, i_1, i_2, i_3 form the basis of $\mathbf{Q}$ and satisfy the following relations:

$$(6)\qquad i_1i_1=i_2i_2=i_3i_3=-1,\quad i_1i_2=-i_2i_1=i_3,$$
$$i_2i_3=-i_3i_2=i_1,\qquad i_3i_1=-i_1i_3=i_2.$$

The conjugate quaternion, $\overline{Q}$, of a quaternion, Q, is defined as

$$(7)\qquad \overline{Q}=q_0-q_1i_1-q_2i_2-q_3i_3$$

One can easily show from (4) and (6) that

$$(9)\qquad i_1^2=i_2^2=\cdots\cdots=i_7^2=-1,$$

$$(10)\qquad i_k\,i_l+i_l\,i_k=0,\ \text{ for }\ l\neq k,\ \ l,k\neq 0$$

where l and k take any value $1, 2, \ldots\ldots 7.$

The multiplication table is as follows.

Table 1.

$AB=c$

$B=$		1	i_1	i_2	i_3	i_4	i_5	i_6	i_7
$A=$	1	1	i_1	i_2	i_3	i_4	i_5	i_6	i_7
	i_1	i_1	-1	i_3	$-i_2$	i_5	$-i_4$	$-i_7$	i_6
	i_2	i_2	$-i_3$	-1	i_1	i_6	i_7	$-i_4$	$-i_5$
	i_3	i_3	i_2	$-i_1$	-1	i_7	$-i_6$	i_5	$-i_4$
	i_4	i_4	$-i_5$	$-i_6$	$-i_7$	-1	$-i_1$	i_2	i_3
	i_5	i_5	i_4	$-i_7$	i_6	i_1	-1	$-i_3$	i_2
	i_6	i_6	i_7	i_4	$-i_5$	$-i_2$	i_3	-1	$-i_1$
	i_7	i_7	$-i_6$	i_5	i_4	$-i_3$	$-i_2$	i_1	-1

From the multiplication table one can easily see that in general the associative law does not hold, i. e.,

$$(12)\qquad (cc')c''\neq c(c'c''),\ \text{ for }\ c,\ c',\ c''\in\mathbf{O}.$$

To characterize this non-associativity one defines an associator $(c,\ c',\ c'')$ by

$$(13)\qquad (cc')c''-c(c'c'')\equiv(c,\ c',\ c'').$$

Then the following identities hold.

(14) $(c, c', c'') = (c', c'', c) = (c'', c, c') = -(c', c, c'')$,

and thus,

(15) $(c, c, c') = 0$.

To avoid any ambiguity in the order of multiplication, parentheses will be used to indicate the order. However, no such ambiguity will arise concerning the product of two quantities, each of which is a quantity raised to different powers, i.e.,

$$A^n = (A^{n-1})A = ((A^{n-2})A)A = \cdots\cdots = \underbrace{AA\cdots\cdots A}_{\text{n times}}$$
$$= (A(A^{n-1})) = A(A(A^{n-2})) = A^{n-r}(A^r)\,.$$

More generally, one can show that for two arbitrary octonions A and B,

(16) $A^l(A^m B) = A(A^{l+m-1}B)$,

and

(17) $A^l B A^m = (A^l B)A^m = A^l(BA^m) = A^{l-r}(A^r B A^s)A^{m-s}$,

for any $l,\ m,\ r,\ s \in \mathbf{Z}^+$.

For any octonion $C = c_0 + c_1 i_1 + \cdots\cdots + c_7 i_7 = (C_1, C_2)$ the conjugate octonion, $\overline{C}$, is defined as

(18) $\overline{C} = c_0 - c_1 i_1 - \cdots\cdots - c_7 i_7 = (\overline{C}_1,\ -C_2)$.

Then it follows that

(19) $\overline{C_1\ C_2} = \overline{C}_2\ \overline{C}_1$.

The norm of an octonion C is defined as

(20) $n(C) = C\overline{C} = \overline{C}C = c_0^2 + c_1^2 + \cdots\cdots + C_7^2$.

3. The regularity conditions.

We now look at the regularity of a function of an octonion variable as discussed in Dentoni and Sce [6].

Define a differential operator, D, by

(21) $D = \sum_{\mu=0}^{7} i_\mu \dfrac{\partial}{\partial x_\mu}\,.$

Then a function, $F(X)$, of an octonion variable, $X = \sum_{\mu=0}^{7} i_\mu x_\mu$, is left regular at X if and only if $F(X)$ satisfies the condition

(22) $DF(X) = 0\,.$

Similarly, a function, $G(X)$, is right regular if and only if

(23) $G(X)D = 0\,.$

Writing the functions in components,

(24) $\quad F(X) = f_0 + f_1 i_1 + \cdots\cdots + f_7 i_7,$

$\quad G(X) = g_0 + g_1 i_1 + \cdots\cdots + g_7 i_7,$

where f_ν and g_ν are real smooth (twice differentiable) functions of x_μ ($\mu = 0, 1, \ldots$ $\ldots 7$), the above regularity conditions (22) and (23) take the following form.

$$(25) \qquad DF = (\partial_0 + \sum_{k=1}^{7} i_k \partial_k)(f_0 + \sum_{j=0}^{7} i_j f_i)$$

$$= \partial_0 f_0 - \sum_{k=1}^{7} \partial_k f_k + \sum_{k=1}^{7} (\partial_0 f_k + \partial_k f_0) i_k + \sum_{(\mu>v,\rho)}^{(1,\cdots 7)} (\partial_\mu f_\nu - \partial_\nu f_\mu) i_\rho,$$

where the summation $\sum\limits_{(\mu>v,\rho)}^{(1,2,\cdots 7)}$ is taken over all μ, ν, $\rho = 1, 2, \ldots\ldots, 7$, and μ, ν, ρ satisfy the relation, $i_\rho = i_\mu i_\nu$, $\mu > \nu$. Writing equation (25) componentwise,

$$\begin{aligned}
(25') \qquad DF = {} & \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7 \\
& + (\partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 - \partial_4 f_5 + \partial_5 f_4 - \partial_6 f_7 + \partial_7 f_6) i_1 \\
& + (\partial_0 f_2 - \partial_1 f_3 + \partial_2 f_0 + \partial_3 f_1 + \partial_4 f_6 + \partial_5 f_7 - \partial_6 f_4 - \partial_7 f_5) i_2 \\
& + (\partial_0 f_3 + \partial_1 f_2 - \partial_2 f \ + \partial_3 f_0 + \partial_4 f_7 - \partial_5 f_6 + \partial_6 f_5 - \partial_7 f_4) i_3 \\
& + (\partial_0 f_4 - \partial_1 f_5 - \partial_2 f_6 - \partial_3 f_7 + \partial_4 f_0 - \partial_5 f_1 - \partial_6 f_2 + \partial_7 f_3) i_4 \\
& + (\partial_0 f_5 + \partial_1 f_4 - \partial_2 f_7 + \partial_3 f_6 - \partial_4 f_1 + \partial_5 f_0 - \partial_6 f_3 + \partial_7 f_2) i_5 \\
& + (\partial_0 f_6 + \partial_1 f_7 + \partial_2 f_4 - \partial_3 f_5 - \partial_4 f_2 + \partial_5 f_3 + \partial_6 f_0 - \partial_7 f_1) i_6 \\
& + (\partial_0 f_7 - \partial_1 f_6 + \partial_2 f_5 + \partial_3 f_4 - \partial_4 f_3 - \partial_5 f_2 + \partial_6 f_1 + \partial_7 f_0) i_7 .
\end{aligned}$$

We now proceed to integrate the right hand side of equation (25') over an eight-dimensional volume v^8. To do so, let ϕ_j be a real continuous functions of the real variables, x_0, x_1, ..., z_7. Using Gauss's theorem over v^8 we have

$$\int_{v^8} (\partial_0 \phi_k + \partial_1 \phi_l + \cdots\cdots + \partial_7 \phi_r) dv^8 = \int_{s^7} (\phi_k dS_0 + \phi_e dS_1 + \cdots\cdots + \phi_r dS_7)$$

We can do this for each of the eight components of equation (25') and thus obtain

$$(26) \qquad \int_{v^8} (DF) dv^8 = \int_{s^7} (dX \cdot F) \equiv F[S^7],$$

where $dX = \sum\limits_{\mu=0}^{7} i_\mu dS_\mu$ is the seven-dimensional hypersurface element of S^7. If all the points on and inside of S^7 are left D-regular, we have

$$(27) \qquad \int_{s^7} (dX \cdot F(X)) = 0, \quad X \in v^8 .$$

This equation is the integral form of the regularity condition for $F(X)$.

4. Functional derivatives and regularity condition.

Let $\bar{S}^7$ be a deformed surface of S^7 of equation (26) in the vicinity of a point, X_0, which lies on S^7.

We define the functional derivative of $F[S^7]$ as

(28) $$\frac{D}{DS^7}F[S^7]=\lim_{\bar{S}_7\to S_7}\frac{F[\bar{S}^7]-F[S^7]}{\int_{\triangle v^8}dv^8},$$

where $\triangle v^8$ is the volume enclosed by the surfaces S^7 and $\bar{S}^7$, and $dv^8=dx_0dx_1\ldots$ $\ldots dx_7$ is the volume element of $\triangle v^8$. Applying Gauss's theorem over the volume $\triangle v^8$, as in equation (26) with $F[\bar{S}^7]-F[S^7]=F[\bar{S}^7-S^7]$, we have,

(29) $$\frac{D}{DS^7}F[S^7]\Big|_{X=X_0}=\lim_{\triangle v^8\to 0}\frac{\int_{\triangle v^8}|DF(X)|_{X=X_0}dv^8}{\int_{\triangle v^8}dv^8}=DF(X_0)\,.$$

Then, if $F(X)$ is left D-regular at $X=X_0$,

$$\frac{D}{DS^7}F[S^7]=0,\quad \text{at } X=X_0\,.$$

This means that the functional $F[S^7]$ defined on the surface is invariant under the deformation of the surface in the vicinity of the point X_0. Furthermore, if $F[S^7]$ is left D-regular at all points in V^8 the value of the functional, defined on the surface S^8, is invariant under continuous deformation of the surface within V^8.

We can extend the above argument to the functional $\Phi[S^7]$ defined by

(30) $$\Phi[S^7]=\int_{S^7}((F(X)dX)\ G(X))\,.$$

When $F(X)$ is left D-regular and $G(X)$ is both side D-regular i.e.,

(31) $DF(X)=0$, and $DG(X)=G(X)D=0$, for all $X\in V^8$,

we can show that (see Appendix)

$$(F(X)D)G(X)=0\,.$$

This implies that the value of the functional $\Phi[S^7]$ is invariant under the deformation of the surface S^7 within the regular domain of V_8 in which the conditions (31) strictly hold.

5. Example of regular functions.

An outstanding example of regular functions can be obtained from regular functions of a complex variable, using the procedure described in Dentoni and Sce[F]; it is an extension of the procedure, for obtaining regular functions of a quaternion variable from that of a complex variable, discussed by Fueter[4].

Let $F(X)$ be a regular function of a complex variable $X=x_0+ix$ $(i=\sqrt{-1})$, which can be written as

$$F(x_0+ix)=u(x_0,\ x)+iv(x_0,\ x),$$

where $u(x_0,\ x)$ and $v(x_0,\ x)$ are real functions and satisfy the Cauchy-Riemann equations:

$$\frac{\partial}{\partial x_0}u(x_0, x) = -\frac{\partial}{\partial x}v(x_0, x),$$

$$\frac{\partial}{\partial x_0}v(x_0, x) = \frac{\partial}{\partial x}u(x_0, x).$$

Now, replacing

$$i = \frac{i_1x_1 + i_2x_2 + \cdots\cdots + i_7x_7}{x},$$

where

$$x = \sqrt{x_1^2 + \cdots\cdots + x_7^2},$$

we can consider X as an octonion variable with

$$i^2 = -1$$

and

$$X = x_0 + x_1i_1 + \cdots\cdots + x_7i_7,$$

for $\qquad x_\mu(\mu=0, \cdots\cdots, 7) \in \mathbf{R}.$

Define

$$G(X) = \square^3 F(X)$$

where

$$\square = \left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \cdots\cdots + \frac{\partial^2}{\partial x_7^2}\right).$$

Then by a straightforward calculation, using the Cauchy-Riemann equations, one can show that $G(X)$ is a both side D-regular function of an octonion variable X, i.e.,

$$DG(X) = G(X)\,D = 0.$$

6. Residue theorem.

The theorem for the function of an octonion variable that corresponds to the residue theorem for the function of a complex variable, was obtained by Dentoni and Sce[6]. In the following we rederive the theorem using functionals which were discussed in the previous section.

In equation (31), choose

(32) $\qquad F(X) = X^{-1}$, and therefore $G(X) = \square^3 X^{-1}$.

Then the following conditions hold:

$$DG(X) = G(X)\,D = 0$$

for all points in $\mathbf{R}^8$, except for $X=0$.

Then from equations (31) and (32), and choosing S^7 as a sphere, K_r^7, of radius r centered at the origin,

$$\Phi[K_r^7] = \int_{K_r^7}((F(X)dX)\,\square^3 X^{-1}) = \text{constant}.$$

The value of $\Phi[K_r^7]$ was calculated and was shown that[6]

$$\lim_{r\to 0} \Phi[K_r{}^7] = \frac{1}{48\pi^4} F(0)\,.$$

Since the functional is a constant and does not change with the radius of the sphere by the angument in the last section, the sphere $K_r{}^7$ can be any closed hypersphere containing the origin. We can easily extend the theorem to the following form

$$\frac{1}{48\pi} \int_{S^7} (F(X)dX)\,(\Box(Z-X)^{-1}) = F(Z)$$

where S^7 is a closed hypersurface which encloses the point, $Z=X$.

7. Regular polynomial functions.

We now construct polynomial functions by defining a generating function

(33) $$F_n(X, \vec{t}) = (\vec{t}\,x_0 - (\vec{t}\cdot\vec{x}))^n = (X\vec{t} - \vec{t}\,X)^n/2^n$$
$$= (t_1(x_0 i_1 - x_1) + t_2(x_0 i_2 - x_2) + \cdots\cdots + t_7(x_0 i_7 - x_7))^n,$$

where $\vec{t} = \sum_{k=1}^{7} t_k i_k$ and $\vec{x} = \sum_{k=1}^{7} x_k i_k$. Define polynomials $P_{n_1 n_2 \cdots n_7}(x_0, x_1, \cdots x_7)$ by expanding $F_n(X, \vec{t})$ in power series as: —

(34) $$F_n(X, \vec{t}) = \sum_{\substack{\sum_{i=1}^{7} ni = n}} n!\, P_{n_1 n_2 \cdots n_7}(x_0, x_1, \dots, x_7)\, t_1{}^{n_1} t_2{}^{n_2} \dots t_7{}^{n_7},$$

where the summation is taken over $n_1, n_2, \cdots\cdots, n_7$ such that $n_1 + n_2 + \cdots\cdots + n_7 = n$. To show that $P_{n_1 n_2 \cdots n_7}$ $(x_0, x_1, \cdots, x_7)$ are left D-regular functions we first prove that $F_n(X, \vec{t}\,)$ satisfies the left D-regular condition. Let

$$K = (\vec{t}\,x_0 - \vec{t}\cdot\vec{x}\,),$$

then

$$DF_n(X, \vec{t}\,) = \sum_{\mu=0}^{7} i_\mu \frac{\partial}{\partial x_\mu} K^n = \frac{\partial}{\partial x_0} K^n + \sum_{k=1}^{7} i_k \frac{\partial}{\partial x_k} K^n\,.$$

Since, for the product of powers of two different octonions K and $\vec{t}$ we need not concern ourselves with the associative law, we simply write the result as: —

$$K^{n-r}\,\vec{t}\,K^{r-1},$$

and thus we have

$$DF_n(X, \vec{t}\,) = \sum_{r=1}^{n} K^{n-r}\vec{t}\,K^{r-1} + \sum_{k=1}^{7} \sum_{r=1}^{n} i_k (K^{n-r}(-t_k)\cdot K^{r-1}).$$

Since K and $\vec{t}$ commute, we have

$$DF_n(X, \vec{t}\,) = \sum_{r=1}^{n} [K^{n-r}\vec{t}\,K^{r-1} - \vec{t}\,K^{n-r}K^{r-1}] = 0\,.$$

From equation (34) and the fact that t_k are independent parameters we see that $P_{n_1 n_2 \cdots n_7}(x_0, x_1, \cdots\cdots, x_7)$ are also left D-regular, i.e.,

(35) $$DP_{n_1 n_2 \cdots n_7}\,(x_0, x_1 \cdots\cdots, x_7) = 0\,.$$

These polynomials are the extension of the Fueter's polynomials of quaternion functions to octonion functions. From the definition (34), the polynomials are expressed explicitly as

(36) $$P_{n_1 n_2 \cdots n_7}(x_0, x_1, \cdots\cdots, x_7) = \frac{1}{n!} \Sigma(i_{k_1}x_0 - x_{k_1})\cdots\cdots(i_{kn}x_0 - x_{kn}),$$

where the summation is taken over all possible permutations of the series $(k_r) = (k_1, \cdots\cdots, k_n)$, where each of $k_1, \cdots\cdots k_n$ can take any value $1, 2, \cdots\cdots, 7$, and that the number of times $1, 2, \cdots\cdots 7$ appears in the series (k_r) is $n_1, n_2, \cdots\cdots, n_7$, respectively. As can be seen from equation (34) the right hand side of equation (36) does not depend on the order of the multiplication of the factors $(i_{kr}x_0 - x_{kr})$, once the order is fixed for each terms hence the parentheses were omitted.

The study on regular functions using these regular polynomials, via tha Fourier representation, the regular exponential functions, and the boundary value problems, will be dealt elsewhere.

Appendix.

To prove that $(F(X)D)G(X)=0$ when $F(X)$ is left D-regular and $G(X)$ is both side D-regular.

Let $G(X)$ be both side D-regular. Then,

(Al) $$DG(X)-G(X)D=0, \text{ i.e,. } \frac{\partial g_\nu}{\partial x_\mu} - \frac{\partial g_\mu}{\partial x_\nu} = 0,$$

and

(A2) $$DG(X)+G(X)D=0, \text{ i.e., } \frac{\partial g_0}{\partial x_k} + \frac{\partial g_k}{\partial x_0} = 0,$$

and

(A3) $$\frac{\partial g_0}{\partial x_0} - \sum_{k=1}^{7} \frac{\partial g_k}{\partial x_k} = 0,$$

for $k, \mu. \nu = 1, 2, \cdots\cdots, 7$.

Then,

$$I \equiv (F(X)D)G(X) = \sum_{\mu=0}^{7} \left(\frac{\partial F}{\partial x_\mu} \cdot i_\mu\right)G(X) + \sum_{\mu=0}^{7} (F \cdot i_\mu)\left(\frac{\partial G}{\partial x_\mu}\right)$$

Since

$$\sum_{\mu=0}^{7} \frac{\partial F}{\partial x_\mu} i_\mu = DF = 0$$

$$I = \sum_{\mu=0}^{7} (F i_\mu)\left(\frac{\partial G}{\partial x_\mu}\right)$$

$$= F\frac{\partial G}{\partial x_0} + \sum_{k=1}^{7}(Fi_k)\left(\frac{\partial g_0}{\partial x_k}\right) + \sum_{k=1}^{7}(F i_k)\left(\frac{\partial g_k}{\partial x_k} i_k\right) + \sum_{\substack{l \neq k \\ l,k \neq 0}} (F i_k)\left(\frac{\partial g_l}{\partial x_k} i_l\right)$$

$$= F\frac{\partial g_0}{\partial x_0} + \sum_{k=1}^{7}\left(F\frac{\partial g_k}{\partial x_0}\right)i_k + \sum_{k=1}^{7}(F i_k)\left(\frac{\partial g_0}{\partial x_k}\right) +$$

$$-\sum_{k=1}^{7} F\frac{\partial g_k}{\partial x_k}+\sum_{\substack{l>k\\ l,k\neq 0}}\left\{(F\,i_k)\frac{\partial g_l}{\partial x_k}i_l+(F\,i_l)\frac{\partial g_k}{\partial x_l}i_k\right\}.$$

Using equations (A2) and (A3),

$$I=\sum_{\substack{l>k\\ l,k\neq 0}}\left\{(F\frac{\partial g_l}{\partial x_k}i_k)\,i_l+(F\frac{\partial g_k}{\partial x_l}i_l)\,i_k\right\}$$

$$=\sum_{\substack{l>k\\ l,k\neq 0\\ \mu=0,1,\cdots,7}}\left\{f_\mu(\frac{\partial g_l}{\partial x_k}((i_\mu i_k)i_l)+\frac{\partial g_k}{\partial x_l}((i_\mu i_l)i_k)\right\}.$$

Since

$$(i_\mu i_k)i_l=-(i_\mu i_l)i_k,\quad \text{for}\quad \mu=0,1,\cdots,7,$$

we have

$$I=\sum_{\substack{l>k\\ l,k\neq 0}} f_\mu\left(\frac{\partial g_l}{\partial x_k}-\frac{\partial g_k}{\partial x_l}\right)((i_\mu i_k)i_l)\,.$$

Then by equation (Al),

$$I=0\,.$$

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