Maximum Principle and Boundary Integral Equation Method in Steady Convective Diffusion Phenomena

Hiroshi NIKI, Michio SAKAKIHARA and Masatoshi IKEUCHI

The authors are with the Graduate School of Science, Okayama University of Science, Ridai-cho 1-1, Okayama-shi 700, Japan

(Received September 20, 1983)

Abstract:

We study on the maximum principle in the boundary integral equation formulation for the convective diffusion equation in a steady state. We also show an example satisfying the maximum principle in incompressible flow case.

1. Introduction.

A boundary integral equation method has been recently presented [2, 3] for a steady solution in the convective diffusion problem. There have been some numerical methods, for example, of the centered difference method, the standard finite element method and so on. All of them have been restricted to domain-type methods. In addition, they give rise to nonphysical oscillation when the convective term is dominant. On the other hand, the upwind methods such as the upwind finite difference method and the upwind finite element method have been developed [4] in order to obtain stable solutions. It is, in general, of the importance for stability that the numerical method satisfies the maximum principle in the convective diffusion problem. Really, some studies on the maximum principle [1, 4] have been known for the domain-type methods. In this paper, we shall consider the maximum principle for theboundary integral equation method. Moreover, we shall show a simple example for incompressible and uniform flow case.

2. Boundary Integral Equation.

Let Ω be the bounded domain in R^m enclosed by the boundary Γ. The convective diffusion equation in a steady state is

$$L[\phi] \overset{d}{=} -\nabla(a(x)\nabla\phi)+\nabla(b(x)\phi)=f(x) \qquad in\ \Omega \qquad (2.\ 1),$$

where $\phi(x)\in H^2(\Omega)$ and $a(x),\ b(x),\ f(x)\in C^1(\Omega)$. The weighted residual statement can be written as

$$\int_{\Omega} L[\phi]\psi d\Omega = \int_{\Omega} f\psi d\Omega \tag{2. 2}$$,

where ψ is some weighting function. By the Green's theorem we have

$$\int_{\Omega} \{L[\phi]\psi - \phi M[\psi]\} d\Omega = \int_{\Gamma} \{-a(\partial\phi/\partial n)\psi + a\phi(\partial\psi/\partial n) + b_n\phi\psi\} d\Gamma \tag{2. 3}$$,

where $M[\cdot]$ is an adjoint operator to the governing operator $L[\cdot]$ defined by

$$M[\psi] \overset{d}{=} -\nabla(a\nabla\psi) + b\nabla\psi \tag{2. 4}$$,

n is the outer normal unit vector to Γ, and b_n is the n component of b. Instead of ψ, the fundamental (or elementary) solution satisfying

$$M[\psi^*] = \delta(x\,;\,y) \tag{2. 5}$$,

where $\delta(x\,;\,y)$ is the Dirac's delta function with the source point $x = (x_1, x_2, \ldots, x_m) \in R^m$. From Eqs. (2. 1) and (2. 5), we can rewrite Eq .(2. 3) to the boundary integral form

$$c(x)\phi(x) - \int_{\Gamma} q_n^*(x\,;\,y)\phi(y) d\Gamma + \int_{\Gamma} \psi^*(x\,;\,y) p_n(y) d\Gamma = \int_{\Omega} \psi^*(x\,;\,y) f(y) d\Omega \tag{2. 6}$$,

where $q_n^*(x\,;\,y)$ and $p_n(y)$ are, respectively, defined by

$$q_n^* = -a(\partial\psi^*/\partial n) - b_n\psi^* \tag{2. 7}$$,

$$p_n = -a(\partial\phi/\partial n) \tag{2. 8}$$,

and $c(x)$ is the weight determined by the solid angle at x.

3. Maximum Principle.

Now let us eamine the maximum principle in the boundary integral equation formulation. We make the following assumptions:

A$_1$) $\nabla b = 0$ *in* Ω,

A$_2$) $\phi \geq 0$ *on* Γ,

and

A$_3$) $\psi^* \geq 0$

Note that the true solution of Eq. (2. 1) is nonnegative over Ω if the assumptions A$_1$), A$_2$) and $f=0$ hold. We also define that

$$\phi_{max} = \max_{y \in \Gamma}\{\phi(y)\} \tag{3. 1}$$,

and

$$\phi_{min} = \min_{y \in \Gamma}\{\phi(y)\} \tag{3. 2}$$.

Then, we have the following lemma:

Lemma 1.

If $x \in \Omega$ *then*

$$\phi_{min} \leq \int_{\Gamma} q_n^*(x;\, y)\phi(y)d\Gamma \leq \phi_{max} \tag{3.3}$$

Proof. Since the source point x is in Ω, $q_n^*(x;\, y)$ has everywhere the same sign for $y \in \Gamma$. We can then take the n such that $q_n^*(x;\, y) \geq 0$. Therefore, we have

$$\phi_{min}\int_{\Gamma} q_n^*(x;\, y)d\Gamma \leq \int_{\Gamma} q_n^*(x;\, y)\phi(y)d\Gamma \leq \phi_{max}\int_{\Gamma} q_n^*(x;\, y)d\Gamma \tag{3.4}$$

In addition, from Eqs. (2. 5) and (2. 7), we know

$$\int_{\Gamma} q_n^*(x;\, y)d\Gamma = \int_{\Omega} \delta(x;\, y)d\Omega = 1 \tag{3.5}$$

Hence *Lemma* 1 can be readily proved.

Let us show the maximum principle for the boundary integral equation method as the following theorem:

Theorem 1 (*Maximum principle*).

Suppose that Ω *is convex and that* $f=0$. *Assume that* $\phi(x)$ *is nontrivial. The solution* $\phi(x)$ *of the boundary integral equation* (2. 6) *satisfies*

$$\int_{\Gamma} q_n^*(x;\, y)\phi(y)d\Gamma - \int_{\Gamma} \psi^*(x;\, y)p_n(y)d\Gamma \leq \phi_{max} \tag{3.6}$$

if $x \in \Omega$ *and* $y \in \Gamma$.

Proof. Suppose that $\phi(x)$ attains to the maximum value ϕ_{max} at some point x_0 in Ω. Then we consider the ball $B(r)$ $(\subset\Omega)$ centered at x_0 where the radius $r = |x_0 - y|$. Then there exists $B(r_0)$ such that $p_n > 0$ (y on $\partial B(r_0)$). From the assumption A_3), we have

$$\int_{\partial B(r_0)} \psi^* p_n(y)d\Gamma \geq 0 \tag{3.7}$$

It also holds that

$$\phi_{max} = \int_{\partial B(r_0)} q_n^*(x_0;\, y)\phi(y)\, dy - \int_{\partial B(r_0)} \psi^*(x_0;\, y)p_n(y)dy.$$

Let us consider the maximum value on $\partial B(r_0)$ denoted by $\bar{\phi}_{max}$. Then from *Lemma* 1, we obtain

$$\int_{\partial B(r_0)} q_n^*(x_0;\, y)\phi(y)\, dy - \int_{\partial B(r_0)} \psi^*(x_0;\, y)p_n(y)dy$$

$$\leq \bar{\phi}_{max} - \int_{\partial B(r_0)} \psi^*(x_0;\, y)p_n(y)dy \leq \phi_{max}.$$

Since $\phi_{max} - \bar{\phi}_{max} \geq 0$, we readily know

$$\int_{\partial B(r_0)} \psi^*(x_0;\, y)p_n(y)dy \leq 0 \tag{3.8}$$

Hence, from Eqs. (3. 7) and (3. 8), it should hold that

$$\int_{\partial B(r_0)} \psi^*(x_0;\, y)p_n(y)dy = 0 \tag{3.9}$$

It can be seen from Eq. (3. 9) that $p_n(y)=0$ in an arbitrary domain $(\subset B(r_0))$, and that ϕ equals to the constant ϕ_{max} in $B(r_0)$. Similarily, we consider another ball $B(r_1)$ centered at some point $(\in B(r_0))$. Then it holds that $p_n(y)=0$ in $B(r_1)$.

Finally, we can show that $\phi=\phi_{max}$ in Ω, because of $\Omega=\cup^k{}_{j=0}B(r_j)$ for the finite k. The above statement contradicts to the assumption, which completes the proof. From the results obtained in the above, we have the following corollaries:

Corollary 1.

Suppose that $f\leq 0$. Then it holds that

$$\phi(x)\leq\phi_{max} \tag{3. 10}$$

Proof. Since $c(x)=1$ for $x\epsilon\Omega$, we know from Eq. (2. 6) that

$$\phi(x)=\int_\Gamma q_n{}^*\phi d\Gamma-\int_\Gamma \psi^* p_n d\Gamma+\int_\Omega \psi^* f d\Omega \tag{3. 11}$$

On the other hand, from the assumption A_3) and $f\leq 0$, Eq. (3. 6) can be rewritten into

$$\int_\Gamma q_n{}^*\phi d\Gamma-\int_\Gamma \psi^* p_n d\Gamma+\int_\Omega \psi^* f d\Omega\leq\int_\Gamma q_n{}^*\phi d\Gamma-\int_\Gamma \psi^* p_n d\Gamma\leq\phi_{max}.$$

Corollary 2.

Suppose that $x\epsilon\Omega$ and $y\epsilon\Gamma$. Then

$$\int_\Gamma \psi^*(x;y)p_n(y)dy\geq 0 \tag{3. 12}$$

4. Example.

Let us show the simple example for three-dimensional case ($\Omega\subset R^3$). Here we assume that $a(x)$ and $b(x)$ are the constants. Note that the above assumption implies the incompressibility of flow A_1) $\nabla b=0$. Then the fundamental solution (or the Green's function in an infinite domain) $\psi^*(x;y)$ is

$$\psi^*(x;y)=\exp[-(b,r)/(2a)-|b||r|/(2a)]/(4\pi a|r|) \tag{4. 1}$$

where $r=x-y$, $x=(x_i)$, $y=(y_i)$, $b=(b_i)$, and $(\cdot,\cdot)$ and $|\cdot|$ denote the inner product, the Euclidean norm, respectively. The derivation of Eq. (4. 1) is stated, as follows: Applying the Liouvile's transformation to Eq. (2. 5), we have

$$-\nabla^2 W^*(|r|)+(b/(2a))^2 W^*(|r|)=\delta(|r|) \tag{4. 2}$$

where

$$\psi^*=\exp[-(b,r)/(2a)]W^*(|r|) \tag{4. 3}$$

The fundamental solution of Eq. (4. 2) is given by

$$W^*=\exp[-|b||r|/(2a)]/(4\pi a|r|) \tag{4. 4}$$

which is known as the Yukawa's potential. Thus we have obtained Eq. (4. 1) from Eqs. (4. 3) and (4. 4). Our fundamental solution $\psi^*(x;y)$ is nonnegative in R^3 for any a, b, x and y. Hence $\psi^*(x;y)$ gives rise to the example which satisfies the statements in the previous chapter. Actually we know many examples in incompressible and unifom flow problems.

5. Concluding Remarks.

In this paper, we present the maximum principle (*Theorem* 1 and *Corollary* 1) for the boundary integral equation formulation and show an example of three-dimensional case. In the steady convective diffusion problem it is an important element for the stability whether the numerical method satisfies the maximum principle. In the discretization of Eq. (2. 6), the result of *Corollary* 2 can be useful for testing the discrete maximum principle. Hence we will be able to employ the analogy to Eq. (3. 12) in the linear systems arising from the numerical method, for example, from the boundary element method, also numerically to examine the stable solution.

References:

[1] Ciarlet, P.G. and Raviart, P.A.: Maximum principle and uniform convergence for the finite element method, Comp. Math. Appl. Mecha. Engin., 2 (1973) 17-31.

[2] Ikeuchi, M. and Onishi, K.: Boundary element solutions to steady convective diffusion equations, Appl. Math. Modelling 7 (1983) 115-118.

[3] Ikeuchi, M., Sakakihara, M. and Onishi, K.: Constant boundary element solution for steady-state convective diffusion equation in three dimension, Trans. IECE Japan, E66 (1983) 373-376.

[4] Kikuchi, F.: Discrete maximum principle and artificial viscosity in finite element approximations to convective diffusion equations, Institute of Space and Aeronautical Science Report, 550 (1977) 153-166.