Potential Kernels of a Biharmonic Space

Hidematu TANAKA

Department of Fundamental Natural Science
Okayama University of Science
Ridai-cho, Okayama 700, Japan

(Received September 20, 1983)

Introduction

Let X be a locally compact Hausdorff space with a countable basis and (X, **H**) be a biharmonic space in the sence of Smyrnêlis [6]. We denote by (X, $\mathbf{H}_j$) ($j=1, 2$) the Bauer's harmonic space associated with (X, **H**) and suppose that (X, $\mathbf{H}_j$) is strong and that 1 is $\mathbf{H}_j$-superharmonic on X ($j=1, 2$). Then there exists a bounded $\mathbf{H}_j$-potential kernel $V^{(j)}$ and the set of all non-negative $\mathbf{H}_j$-hyperharmonic functions, the set of all lower semi-continuous $V^{(j)}$-dominant functions and the set of all $V^{(j)}$-excessive functions coinside ($j=1, 2$) (c. f. [1]). Let K be the composing kernel of (X, **H**) and set $W=KV^{(2)}$. We define ($V^{(1)}$, W, $V^{(2)}$)-dominant couples and ($V^{(1)}$, W, $V^{(2)}$)-excessive couples associated with the system of potential kernels ($V^{(1)}$, W, $V^{(2)}$) and we shall show that the set of all non-negative **H**-hyperharmonic couples, the set of all ($V^{(1)}$, W, $V^{(2)}$)-dominant couples and the set of all ($V^{(j)}$, W, $V^{(2)}$)-excessive couples coinside. Theorem 12 in [3] contains partially this result. In this note we shall use a slight different method from [3] to show the above result.

1. Biharmonic spaces

Let X be a locally compact Hausdorff space with a countable basis. For an open set $U \neq \phi$ on X, we denote by **C**(U) the real vector space of finite continuous functions on U. An element (h_1, h_2) in **C**(U)×**C**(U) is called compatible if $h_1=0$ on an open subset U′ of U implies $h_2=0$ on U′. Let **H** be an application U→**H**(U), where **H**(U) is a real vector subspace of compatible couples in **C**(U)×**C**(U).

A relatively compact open set ω is called **H**-regular if for any couple (f_1, f_2) of finite continuous functions on the boundary $\partial\omega$ of ω, there exists a unique (h_1, h_2)$\in$**H**(ω) such that:

(i) $\lim_{x \to a} h_j(x) = f_j(a)$ for any $a \in \partial\omega$ $(j=1,2)$;

(ii) $f_j \geqq 0$ $(j=1, 2)$ implies $h_1 \geqq 0$ and $f_2 \geqq 0$ implies $h_2 \geqq 0$.

For an **H**-regular set ω, there exists a unique system $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ of positive Radon measures on $\partial\omega$ such that

$$h_1(x) = \int f_1 d\mu_x^\omega + \int f_2 d\nu_x^\omega, \ h_2(x) = \int f_2 d\lambda_x^\omega.$$

We say that (X, **H**) is a biharmonic space in the sence of Smyrnélis [6] if it satisfies the following four axioms.

Axiom I. H is a sheaf on X.

For an open set U in X, an element in **H**(U) is called **H**-harmonic on U.

Axiom II. H-regular open sets form a basis of X.

A couple (v_1, v_2) of functions on an open set U is called **H**-hyperharmonic on U if

(i) v_j is lower semi-continuous and $>-\infty$ on U $(j=1, 2)$,

(ii) $v_1(x) \geqq \int v_1 d\mu_x^\omega + \int v_2 d\nu_x^\omega$ and $v_2(x) \geqq \int v_2 d\lambda^\omega{}_x$ for any **H**-regular neighborhood ω of x with $\bar{\omega} \subset$ U.

An **H**-hyperharmonic couple (s_1, s_2) on U is called **H**-superharmonic on U if $s_j < +\infty$ on a dense subset of U$(j=1, 2)$ and an **H**-superharmonic couple (p_1, p_2) on X is called an **H**-potential on X if $p_j \geqq 0$ and, for any $(h_1, h_2) \in \mathbf{H}(X)$, $h_j=0$ so far as $0 \leqq h_j \leqq p_j$ $(j=1, 2)$. The set of all **H**-potentials on X is denoted by **P**(X). For an open set U, denote by **H***(U) the set of all **H**-hyperharmonic couples on U and put $\mathbf{H}_1^*(U) = \{v_1 : (v_1, 0) \in \mathbf{H}^*(U)\}$, $\mathbf{H}_2^*(U) = \{v_2 : (v_1, v_2) \in \mathbf{H}^*(U)$ for some $v_1\}$ and $\mathbf{H}_j(U) = \mathbf{H}_j^*(U) \cap [-\mathbf{H}_j^*(U)]$ $(j=1, 2)$.

Axiom III. (i) $\mathbf{H}_j^*$ (X) separates the points of X linearly $(j=1, 2)$.

(ii) On each relatively compact open set U there exists a strictly positive $h_j \in \mathbf{H}_j(U)$ $(j=1, 2)$.

Axiom IV. If $\{h_j^{(n)}\}_n$ is an increasing sequence of functions in $\mathbf{H}_j(U)$ and $\sup_n h_j^{(n)} < +\infty$ on a dense subset of an open set U, then $\sup_n h_j^{(n)} \in \mathbf{H}_j(U)$ $(j=1, 2)$.

Set $\mathbf{H}_j = \{\mathbf{H}_j(U)\}_{U: \text{open}}$ It is shown by Theorem 1.29 in [6] that (X, $\mathbf{H}_j$) $(j=1, 2)$ is a Bauer's harmonic space. We call (X, $\mathbf{H}_j$) $(j=1, 2)$ the harmonic space associated with (X, **H**). The set of all $\mathbf{H}_j$-superharmonic functions (resp. $\mathbf{H}_j$-potentials) on X is denoted by $\mathbf{S}_j(X)$ (resp. $\mathbf{P}_j(X)$) $(j=1, 2)$.

2. Composing kernels

Let $(X, \mathbf{H})$ be a biharmonic space and $(X, \mathbf{H}_j)$ $(j=1, 2)$ be the harmonic space associated with $(X, \mathbf{H})$. Let $\mathbf{U}$ be an open covering of X consisting of $\mathbf{H}$-regular sets ω on which there exists a strictly positive $\mathbf{H}_1$-potential and $\int d\lambda_x^\omega > 0$ on ω (c.f. [4]). A family $(p_\omega)_{\omega \in \mathbf{U}}$ of $\mathbf{H}_1$-potentials p_ω on ω is called consistent if $p_{\omega_1} - p_{\omega_2}$ is $\mathbf{H}_1$-harmonic on $\omega_1 \cap \omega_2$ for any ω_1 and ω_2 in $\mathbf{U}$. For an $\mathbf{H}_1$-potential p_ω on $\omega \in \mathbf{U}$, we denote by K_{p_ω} the potential kernel associated with p_ω (c.f. Theorem 3.2 in [1]).

Definition 1. The family $(K_{p_\omega})_{\omega \in \mathbf{U}}$ of potential kernels associated with a consistent family $(p_\omega)_{\omega \in \mathbf{U}}$ of $\mathbf{H}_1$-potentials is called the family of composing kernels of $(X, \mathbf{H})$ if for any $\omega \in \mathbf{U}$ and for any $f \in \mathbf{C}(\partial\omega)$,

$$\int f d\nu_x^\omega = \int \left(\int f d\lambda_y^\omega\right) K_{p_\omega}(x, dy)$$

on ω.

By Theorem 3.5 in [2] we have

Lemma 1. *There exists a unique family of composing kernels* $(K_{p_\omega})_{\omega \in \mathbf{U}}$ *of* $(X, \mathbf{H})$.

By the consistency of $(p_\omega)_{\omega \in \mathbf{U}}$ we shall show the following lemma analogously to the proof of Proposition 2.7 in [5].

Lemma 2. *If there exists a strictly positive* $\mathbf{H}_1$*-potential on* X, *then there exists a unique continuous* $\mathbf{H}_1$*-potential* p *on* X *such that* $(K_{p_\omega})_{\omega \in \mathbf{U}}$ *is the family of composing kernels of* $(X, \mathbf{H})$, *where* $p_\omega = p - \int p d\mu^\omega$. *for any* $\omega \in \mathbf{U}$.

The potential kernel associated with the above continuous $\mathbf{H}_1$-potential p is denoted by K and called the composing kernel of $(X, \mathbf{H})$. The above family $(K_{p_\omega})_{\omega \in \mathbf{U}}$ is called the family of composing kernels associated with K.

Let v_2 be in $\mathbf{H}_2^*(X)$ and $v_2 \geqq 0$. We set

$$\mathbf{E}_{v_2} = \{w_1 : w_1 \geqq 0 \text{ and } (w_1, v_2) \in \mathbf{H}^*(X)\}$$

and $v_1 = \inf \mathbf{E}_{v_2}$ Then $v_1 \in \mathbf{E}_{v_2}$ by Lemma 11.6 in [6]. We call this function v_1 the pure hyperharmonic function of order 2 associated with v_2 (c.f. [6] and [7]). Then we have the following lemma (c.f. Lemma 11.8 in [6]).

Lemma 3. *Suppose that* $1 \in \mathbf{S}_2(X)$. *Let* v_2 *be in* $\mathbf{H}_2^*(X)$ *with* $v_2 \geqq 0$ *and* v_1 *be the pure hyperharmonic function of order 2 associated with* v_2 *and let* $(K_{p_\omega})_{\omega \in \mathbf{U}}$ *be the family of composing kernels of* $(X, \mathbf{H})$. *Then*

$$v_1(x) = \int v_1 d\mu_x^{\omega} + \int v_2(y) K_{p_\omega}(x, dy)$$

for any $\omega \in \mathbf{U}$ *and* $x \in \omega$.

Proof. Let $\omega \in \mathbf{U}$. We put

$$w_1(x) = \begin{cases} \inf \left(v_1(x), \int v_1 d\mu_x^{\omega} + \int v_2(y) K_{p_\omega}(x, dy)\right) & x \in \omega \\ v_1(x) & x \in X - \omega \end{cases}$$

and $w_2(x) = v_2(x)$ on X. It suffices to show that $(w_1, w_2) \in \mathbf{H}^*(X)$. Let $\{f_j^{(n)}\}_n$ be an increasing sequence of non-negative continuous functions on X such that $\lim_{n\to\infty} f_j^{(n)} = v_j$ $(j=1, 2)$. For any $\varepsilon > 0$, we put

$$u_{n,\varepsilon}(x) = \int f_1^{(n)} d\mu_x^{\omega} + \int \left(f_2^{(n)}(y) - \varepsilon \int d\lambda_y^{\omega}\right) K_{p_\omega}(x, dy)$$

and show that $v_1 - u_{n,\varepsilon} \in \mathbf{H}_1^*(X)$. Let $x \in \omega$ and $\alpha = f_2^{(n)}(x) / \int d\lambda_x^{\omega} - \varepsilon/2$. Then $U = \{y \in \omega : f_2^{(n)}(y) - \varepsilon \int d\lambda_y^{\omega} < \alpha \int d\lambda_y^{\omega} < f_2^{(n)}(y)\}$ is an open neighborhood of x. Let ω' be an $\mathbf{H}$- regular set with $x \in \omega' \subset \overline{\omega}' \subset U$. Then

$$\begin{aligned} &(v_1 - u_{n,\varepsilon}) - \int (v_1 - u_{n,\varepsilon}) d\mu_\cdot^{\omega'} \\ &= v_1 - \int v_1 d\mu_\cdot^{\omega'} - \int \left(f_2^{(n)}(y) - \varepsilon \int d\lambda_y^{\omega}\right) K_{p_{\omega'}}(\cdot, dy) \\ &\geqq v_1 - \int v_1 d\mu_\cdot^{\omega'} - \int \left(\alpha \int d\lambda_y^{\omega}\right) K_{p_{\omega'}}(\cdot, dy) \\ &= v_1 - \int v_1 d\mu_\cdot^{\omega'} - \int \left(\alpha \int d\lambda_y^{\omega}\right) d\nu_\cdot^{\omega'}(y) \\ &\geqq v_1 - \int v_1 d\mu_\cdot^{\omega'} - \int f_2^{(n)}(y) d\nu_\cdot^{\omega'}(y) \\ &\geqq \int (v_2(y) - f_2^{(n)}(y)) d\nu_\cdot^{\omega'}(y) \geqq 0. \end{aligned}$$

Hence we have that $v_1 - u_{n,\varepsilon} \in \mathbf{H}_1^*(\omega)$.
Letting $\varepsilon \to 0$, $v_1 - \int f_1^{(n)}(y) d\mu_\cdot^{\omega}(y) - \int f_2^{(n)} K_{p_\omega}(\cdot, dy) \in \mathbf{H}_1^*(\omega)$.

On the other hand, $\int f_2^{(n)}(y) K_{p_\omega}(\cdot, dy)$ being a continuous $\mathbf{H}_1$-potential on ω, $v_1 - \int f_1^{(n)} d\mu_\cdot^{\omega} - \int f_2^{(n)}(y) K_{p_\omega}(\cdot, dy) \geqq 0$ on ω and therefore letting $n \to \infty$, we conclude that $v_1 \geqq \int v_1 d\mu_\cdot^{\omega} + \int v_2(y) K_{p_\omega}(\cdot, dy)$ on ω. Since $(w_1, w_2) \in \mathbf{H}^*(\omega)$ and $\lim_{\omega \ni x \to z} w_1(x) \geqq v_1(z)$ for any $z \in \partial\omega$, $(w_1, w_2) \in \mathbf{H}^*(X)$ by Proposition 1.21 in [6]. This completes the proof.

From now on we shall suppose that $(X, \mathbf{H}_j)$ is strong that is there exists a strictly positive $\mathbf{H}_j$-potential on X $(j=1, 2)$. Let K be the composing kernel of $(X, \mathbf{H})$ and $(K_{p_\omega})_{\omega \in \mathbf{U}}$ be the family of composing kernel associated with K (c.f. Lemma 2). We put

$$\mathbf{P}_0(X)=\left\{\begin{array}{r}(p_1,p_2)\in\mathbf{P}(X):p_1 \text{ is a pure hyperharmonic function}\\ \text{of order 2 associated with } p_2\end{array}\right\}$$

Then we have

Lemma 4. *Suppose that* $1\in\mathbf{S}_2(X)$. *If* $(p_1,p_2)\in\mathbf{P}_0(X)$, *then* $p_1=Kp_2$.

Proof. Since $(X,\mathbf{H}_2)$ is strong and $1\in\mathbf{S}_2(X)$, there exists an increasing sequence $\{p_2^{(n)}\}_n$ of bounded continuous $\mathbf{H}_2$-potentials on X such that $\lim_{n\to\infty}p_2^{(n)}=p_2$. The couple $(p_1,p_2^{(n)})$ being in $\mathbf{P}(X)$, there exists $p_1^{(n)}\in\mathbf{P}_1(X)$ such that $(p_1^{(n)},p_2^{(n)})\in\mathbf{P}_0(X)$. Since $Kp_2^{(n)}\in\mathbf{P}_1(X)$, we have that $p_1^{(n)}=Kp_2^{(n)}$ by Lemma 3. Hence $p_1\geqq Kp_2$. On the other hand, $(Kp_2,p_2)\in\mathbf{H}^*(\omega)$ for any $\omega\in U$, because for any $\mathbf{H}$-regular set ω' with $\omega'\subset\overline{\omega}'\subset\omega$,

$$\begin{aligned}Kp_2\geqq Kp_2^{(n)}&=\int Kp_2^{(n)}\,d\mu_\cdot^{\omega'}+\int p_2^{(n)}(y)K_{p_{\omega'}}(\cdot,dy)\\&\geqq\int Kp_2^{(n)}\,d\mu_\cdot^{\omega'}+\int\Big(\int p_2^{(n)}\,d\lambda_y^{\omega'}\Big)K_{p_{\omega'}}(\cdot,dy)\\&=\int Kp_2^{(n)}\,d\mu_\cdot^{\omega'}+\int p_2^{(n)}\,d\nu_\cdot^{\omega'},\end{aligned}$$

and $Kp_2\geqq\int Kp_2 d\mu_\cdot^{\omega'}+\int p_2 d\nu_\cdot^{\omega'}$. Hence $(Kp_2,p_2)\in\mathbf{H}^*(X)$ and so $Kp_2\geqq p_1$. This completes the proof.

3. Potential kernels

Let $(X,\mathbf{H})$ be a biharmonic space and $(X.\ \mathbf{H}_j)$ $(j=1,2)$ be the harmonic space associated with $(X,\mathbf{H})$. In this section we shall suppose that the following axiom is satisfied.

Axiom V. $(X,\mathbf{H}_j)$ is strong and $1\in\mathbf{S}_j(X)$ $(j=1,2)$.

We denote by $T(p_1,p_2)$ (resp. $T(p_j)(j=1,2)$) the biharmonic support of $(p_1,p_2)\in\mathbf{P}(X)$ (resp. $\mathbf{H}_j$-harmonic support of $p_j\in\mathbf{P}_j(X)$ $(j=1,2)$). Then we see that $T(p_2)\subset T(p_1,p_2)\subset T(p_1)$ and $T(p_2)=T(p_1,p_2)$ if $(p_1,p_2)\in\mathbf{P}_0(X)$. We denote by $\mathbf{M}^+(X)$ (resp. $\mathbf{M}_b^+(X)$) the set of all (resp. all bounded) non-negative Borel measurable functions on X.

The following theorem is evident by Theorem 3.2 in [1] and Lemma 4.

Theorem 1. *Let* (p_1,p_2) *be a continuous couple in* $\mathbf{P}_0(X)$. *Then there exists a unique couple of kernels* (W,V) *such that*

(i) $(W1,V1)=(p_1,p_2)$,

(ii) *for any* $f\in\mathbf{M}_b^+(X)$, (Wf,Vf) *is a continuous couple in* $\mathbf{P}_0(X)$ *with* $T(Wf,Vf)\subset\{f>0\}$.

Since $(X, \mathbf{H}_j)$ is strong and $1 \in \mathbf{S}_j(X)$, there exists an $\mathbf{H}_j$-potential kernel $V^{(j)}$ satisfing the following properties $(j=1, 2)$ (c.f. [1]).

(i) The kernel $V^{(j)}$ is bounded and has a submarkovian resolvent $(V_\lambda^{(j)})_{\lambda \geq 0}$ such that $V_0^{(j)} = V^{(j)}$ $(j=1, 2)$.

(ii) Let $\mathbf{H}_j^*(X)^+$ be the set of all non-negative $\mathbf{H}_j$-hyperharmonic functions on X, $\mathbf{D}_s(V^{(j)})$ be the set of all lower semi-continuous $V^{(j)}$-dominant functions and $\mathbf{E}(V^{(j)})$ be the set of all $V^{(j)}$-excessive functions. Then the following three sets of functions coincide:

$$\mathbf{H}^*_j(X)^+ = \mathbf{D}_s(V^{(j)}) = \mathbf{E}(V^{(j)}) \quad (j=1, 2).$$

Since $V^{(2)}$ is bounded, $(KV^{(2)}1, V^{(2)}1)$ is a continuous couple in $\mathbf{P}_0(X)$. Hence by Theorem 1 there exists a unique couple of kernels $(W, V^{(2)})$ such that $W = KV^{(2)}$.

Definition 2. A couple $(d_1, d_2) \in \mathbf{M}^+(X) \times \mathbf{M}^+(X)$ is called $(V^{(1)}, W, V^{(2)})$-dominant if for all $f_1, f_2, g_1, g_2 \in \mathbf{M}^+(X)$ the following implication holds:

$$\begin{cases} d_1 + V^{(1)} f_1 + W f_2 \geqq V^{(1)} g_1 + W g_2 & \text{on } \{g_1 > 0\}, \\ d_2 + V^{(2)} f_2 \geqq V^{(2)} g_2 & \text{on } \{g_2 > 0\} \end{cases}$$

$$\Longrightarrow \begin{cases} d_1 + V^{(1)} f_1 + W f_2 \geqq V^{(1)} g_1 + W g_2 & \text{on } X, \\ d_2 + V^{(2)} f_2 \geqq V^{(2)} g_2 & \text{on } X. \end{cases}$$

The set of all $(V^{(1)}, W, V^{(2)})$-dominant (resp. lower semi-continuous $(V^{(1)}, W, V^{(2)})$-dominant) couples is denoted by $\mathbf{D}(V^{(1)}. W, V^{(2)})$ (resp. $\mathbf{D}_s(V^{(1)}, W, V^{(2)})$). Let $\mathbf{H}^*(X)^+$ be the set of all non-negative H-hyperharmonic couples on X. Then we have

Lemma 5. $\mathbf{H}^*(X)^+ \subset \mathbf{D}_s(V^{(1)}, W, V^{(2)})$.

Proof. Let $(u_1, u_2) \in \mathbf{H}^*(X)^+$. Assume that

$$\begin{cases} u_1 + V^{(1)} f_1 + W f_2 \geqq V^{(1)} g_1 + W g_2 & \text{on } \{g_1 > 0\}, \\ u_2 + V^{(2)} f_2 \geqq V^{(2)} g_2 & \text{on } \{g_2 > 0\} \end{cases}$$

holds for all $f_1, f_2, g_1, g_2 \in \mathbf{M}^+(X)$. Since $u_2 \in \mathbf{H}_2^*(X)^+$, $u_2 \in \mathbf{D}_s(V^{(2)})$ by Lemma 3.5 in [1] and so $u_2 + V^{(2)} f_2 \geqq V^{(2)} g_2$ on X. Let $\{p_2^{(n)}\}$ be an increasing sequece of bounded continuous $\mathbf{H}_2$-potentials such that $\lim_{n\to\infty} p_2^{(n)} = u_2$. Then $(Kp_2^{(n)}, p_2^{(n)})$ being in $\mathbf{P}_0(X)$, $u_1 \geqq Kp_2^{(n)}$ and so $u_1 \geqq Ku_2$. Hence $u_1 + W(f_2 - g_2) = u_1 + KV^{(2)}(f_2 - g_2) \geqq u_1 - Ku_2 \geqq 0$. Therefore it suffices to show that $u_1 + W(f_2 - g_2) \in \mathbf{H}_1^*(X)$. Let $\omega \in \mathbf{U}$ and $v = u_1 + W(f_2 - g_2)$. Then for any $\mathbf{H}$-regular set ω' with $\omega' \subset \overline{\omega}' \subset \omega$, similarly to the proof of Lemma 3 we have

$$v - \int v d\mu_\cdot^{\omega'} = u_1 - \int u_1 d\mu^{\omega'} - \int V^{(2)}(g_2 - f_2)(y) K_{p_{\omega'}}(\cdot, dy)$$

$$\geqq u_1 - \int u_1 d\mu_\cdot^{\omega'} - \int u_2(y) K_{p\omega'}(\cdot, dy)$$
$$\geqq 0.$$

Hence $v \in \mathbf{H}_1{}^*(\omega)$ and so $v \in \mathbf{H}_1{}^*(X)$. This completes the proof.

By Proposition 9 in [3] we have the following

Lemma 6. *There exists a unique family of kernels* $(W_\lambda)_{\lambda > 0}$ *such that*

$$(*) \quad W_\lambda - W_\mu = (\mu - \lambda)(V_\lambda{}^{(1)} W_\mu + W_\lambda V_\mu{}^{(2)})$$
$$= (\mu - \lambda)(V_\mu{}^{(1)} W_\lambda + W_\mu V_\lambda{}^{(2)})$$

for any λ, $\mu > 0$ *and* $\sup_{\lambda > 0} W_\lambda = W$.

We denote by $\mathbf{D}(V^{(j)})$ the set of all $V^{(j)}$-dominant functions (c.f. [1]). Then we have

Theorem 2. *If* $(d_1, d_2) \in \mathbf{D}(V^{(1)}, W, V^{(2)})$, *then*

(i) $d_1 \in \mathbf{D}(V^{(1)})$, $d_2 \in \mathbf{D}(V^{(2)})$;

(ii) $d_1 \geqq \lambda V_\lambda{}^{(1)} d_1 + \lambda W_\lambda d_2$, $d_2 \geqq \lambda V_\lambda{}^{(2)} d_2$ *for any* $\lambda > 0$;

(iii) $d_1 - K\hat{d}_2 \in \mathbf{D}(V^{(1)})$, *where* $\hat{d}_2 = \lim_{\lambda \to \infty} \lambda V_\lambda{}^{(2)} d_2$.

Proof. Since (i) is evident by the definition, $d_2 \geqq \lambda V_\lambda{}^{(2)} d_2$ holds for any $\lambda > 0$ and so there exists the excessive regularization $\hat{d}_2$ of d_2. Let $f = d_1 - \lambda V_\lambda{}^{(1)} d_1 - \lambda W_\lambda d_2$, $g = \lambda(I - \lambda V_\lambda{}^{(2)}) d_2$, $f_1 = \max(-f, 0)$ and $f_2 = \max(f, 0)$. Then by the equation (*) of Lemma 6, we have

$$d_1 - f + V^{(1)}(\lambda f_1) = V^{(1)}(\lambda f_2) + Wg.$$

Hence

$$\begin{cases} d_1 + V^{(1)}(\lambda f_1) \geqq V^{(1)}(\lambda f_2) + Wg & \text{on } \{\lambda f_2 > 0\}, \\ d_2 + V^{(2)} 0 \geqq V^{(2)} g & \text{on } \{g > 0\}. \end{cases}$$

Since $(d_1, d_2) \in \mathbf{D}(V^{(1)}, W, V^{(2)})$, we have

$$d_1 + V^{(1)}(\lambda f_1) \geqq V^{(1)}(\lambda f_2) + Wg$$

on X and so

$$d_1 \geqq V^{(1)}(\lambda(d_1 - \lambda V_\lambda{}^{(1)} d_1 - \lambda W_\lambda d_2) + W(\lambda(I - \lambda V_\lambda{}^{(2)}) d_2)$$
$$= \lambda V_\lambda{}^{(1)} d_1 + \lambda W_\lambda d_2$$

for any $\lambda > 0$. By this inequality and the resolvent equation of $V^{(j)}$ $(j = 1, 2)$, we have

$$d_1 \geqq (I + \lambda V^{(1)}) \lambda W_\lambda d_2$$
$$= W(\lambda(I - \lambda V_\lambda{}^{(2)}) d_2)$$
$$= K(\lambda V_\lambda{}^{(2)} d_2)$$

and therefore, letting $\lambda \to \infty$, we have that $d_1 \geqq K\hat{d}_2$. Assume that

$$d_1 - K\hat{d}_2 + V^{(1)}\varphi_1 \geqq V^{(1)}\varphi_2 \text{ on } \{\varphi_2 > 0\}$$

for any $\varphi_1, \varphi_2 \in \mathbf{M}^+(X)$. Then

$$d_1 + V^{(1)}\varphi_1 \geqq V^{(1}\varphi_2 + W(\lambda(I - \lambda V_\lambda^{(2)})d_2) \text{ on } \{\varphi_2 > 0\}$$

and so we have

$$\begin{cases} d_1 + V^{(1)}\varphi_1 + W(\lambda^2 V_\lambda^{(2)} d_2) \geqq V^{(1)}\varphi_2 + W(\lambda d_2) & \text{on } \{\varphi_2 > 0\}, \\ d_2 + V^{(2)}(\lambda^2 V_\lambda^{(2)} d_2) \geqq V^{(2)}(\lambda d_2) & \text{on } \{\lambda d_2 > 0\}. \end{cases}$$

Since $(d_1, d_2) \in \mathbf{D}(V^{(1)}, W, V^{(2)})$, we have

$$\begin{aligned} d_1 + V^{(1)}\varphi_1 &\geqq V^{(1)}\varphi_2 + W(\lambda(I - \lambda V_\lambda^{(2)})d_2) \\ &= V^{(1)}\varphi_2 + K(\lambda V_\lambda^{(2)} d_2) \end{aligned}$$

on X. Letting $\lambda \to \infty$, we have $d_1 - Kd_2 + V^{(1)}\varphi_1 \geqq V^{(1)}\varphi_2$ on X and so $d_1 - K\hat{d}_2 \in \mathbf{D}(V^{(1)})$. This completes the proof.

Definition 3. A couple $(e_1, e_2) \in \mathbf{M}^+(X) \times \mathbf{M}^+(X)$ is called $(V^{(1)}, W, V^{(2)})$-excessive if for any $\lambda > 0$

$$e_1 \geqq \lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2, \quad e_2 \geqq \lambda V_\lambda^{(2)} e_2$$

and

$$e_1 = \lim_{\lambda \to \infty} (\lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2), \quad e_2 = \lim_{\lambda \to \infty} \lambda V_\lambda^{(2)} e_2.$$

The set of all $(V^{(1)}, W, V^{(2)})$-excessive couples is denoted by $\mathbf{E}(V^{(1)}, W, V^{(2)})$ Then we have

Lemma 7. $\mathbf{H}^*(X)^+ \subset \mathbf{E}(V^{(1)}, W, V^{(2)})$.

Proof. Let (u_1, u_2) be in $\mathbf{H}^*(X)^+$ and $\{p_2^{(n)}\}_n$ be an increasing sequence of bounded continuous $\mathbf{H}_2$-potentials with $\lim_{n\to\infty} p_2^{(n)} = u_2$. Since $u_1 - Kp_2^{(n)} \in \mathbf{H}_1^*(X)^+ = \mathbf{E}(V^{(1)})$, $(u_1 - Kp_2^{(n)}, 0) \in \mathbf{E}(V^{(1)}, W, V^{(2)})$. We shall show that $(Kp_2^{(n)}, p_2^{(n)}) \in \mathbf{E}(V^{(1)}, W, V^{(2)})$. For any $\lambda > 0$,

$$\begin{aligned} \lambda V_\lambda^{(1)} Kp_2^{(n)} + \lambda W_\lambda p_2^{(n)} &= \lambda V_\lambda^{(1)} Kp_2^{(n)} + (I - \lambda V_\lambda^{(1)})K(\lambda V^{(2)}(I - \lambda V_\lambda^{(2)})p_2^{(n)}) \\ &= K(\lambda V_\lambda^{(2)} p_2^{(n)}) + \lambda V_\lambda^{(1)} K(p_2^{(n)} - \lambda V_\lambda^{(2)} p_2^{(n)}) \\ &\leqq K(\lambda V_\lambda^{(2)} p_2^{(n)}) + K(p_2^{(n)} - \lambda V_\lambda^{(2)} p_2^{(n)}) \\ &= Kp_2^{(n)}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} 0 &\leqq Kp_2^{(n)} - \lambda V_\lambda^{(1)} Kp_2^{(n)} - \lambda W_\lambda p_2^{(n)} \\ &= (I - \lambda V_\lambda^{(1)})K(p_2^{(n)} - \lambda V_\lambda^{(2)} p_2^{(n)}) \\ &\leqq (I - \lambda V_\lambda^{(1)})Kp_2^{(n)} \to 0 \ (\lambda \to \infty), \end{aligned}$$

we know that $(Kp_2^{(n)}, p_2^{(n)}) \in \mathbf{E}(V^{(1)}, W, V^{(2)})$. Hence $(u_1, p_2^{(n)})$ is in $\mathbf{E}(V^{(1)}, W, V^{(2)})$. Letting $n \to \infty$, we conclude that (u_1, u_2) is in $\mathbf{E}(V^{(1)}, W, V^{(2)})$. This completes the proof.

We shall show the following theorem (c.f. Theorem 12 in [3]).

Theorem 3. $\mathbf{H}^*(X)^+ = \mathbf{D}_s(V^{(1)}, W, V^{(2)}) = \mathbf{E}(V^{(1)}, W, V^{(2)})$.

Proof. By Lemmas 5 and 7 it suffices to show that $\mathbf{D}_s(V^{(1)}, W, V^{(2)}) \subset \mathbf{E}(V^{(1)}, W, V^{(2)}) \subset \mathbf{H}^*(X)^+$. Let $(d_1, d_2) \in \mathbf{D}_s(V^{(1)}, W, V^{(2)})$. Then by Theorem 2, $d_j \in \mathbf{D}_s(V^{(j)})$ $(j=1, 2)$ and $d_1 \geqq \lambda V_\lambda^{(1)} d_1 + \lambda W_\lambda d_2$ for any $\lambda > 0$. Hence $d_j \in \mathbf{E}(V^{(j)})$ $(j=1, 2)$ and $d_1 = \lim_{\lambda \to \infty} (\lambda V_\lambda^{(1)} d_1 + \lambda W_\lambda d_2)$. Therefore we know that $(d_1, d_2) \in \mathbf{E}(V^{(1)}, W, V^{(2)})$.

To show that $\mathbf{E}(V^{(1)}, W, V^{(2)}) \subset \mathbf{H}^*(X)^+$, let (e_1, e_2) be in $\mathbf{E}(V^{(1)}, W, V^{(2)})$. By the equality (*) of Lemma 6 we see that

$$\lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2 \geqq \mu V_\mu^{(1)} e_1 + \mu W_\mu e_2, \quad \lambda V_\lambda^{(2)} e_2 \geqq \mu V_\mu^{(2)} e_2$$

for any $\lambda \geqq \mu$. On the other hand, we have

$$(\lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2, \lambda V_\lambda^{(2)} e_2)$$
$$= (V^{(1)} (\lambda(e_1 - \lambda V_\lambda^{(1)} e_1 - \lambda W_\lambda e_2)), 0) + (W(\lambda(I - \lambda V_\lambda^{(2)}) e_2), V^{(2)} (\lambda(I - \lambda V_\lambda^{(2)}) e_2)),$$

and so $(\lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2, \lambda V_\lambda^{(2)} e_2) \in \mathbf{H}^*(X)^+$. Since (e_1, e_2) in the limit of an increasing sequence of couples $\{(\lambda V_\lambda^{(1)} e_1 + \lambda W_\lambda e_2, \lambda V_\lambda^{(2)} e_2)\}_\lambda$, we conclude that $(e_1, e_2) \in \mathbf{H}^*(X)^+$. This completes the proof.

References

[1] H. Bauer: Harmonic spaces and associated Markov processes. Potential theory (C.I.M.E.) (1970), 23–69.

[2] A. Boukricha: Axiomatique des espaces biharmoniques. to appear

[3] N. Bouleau: Espaces biharmoniques et couplage de processus de Markov. J. Math. pures et appl. 58, (1979), 187–240.

[4] C. Constantinescu and A. Cornea: Potential theory of harmonic spaces. Springer Verlag (1972).

[5] W. Hansen: Perturbation of harmonic spaces. Math. Ann. 251, (1980), 111–122.

[6] E. P. Smyrnélis: Axiomatique des fonctions biharmoniques. (I. Ann. Inst. Fourier, 25, (1975), 35–67), (II. Ann. Inst. Fourier, 26, (1976), 1–47).

[7] E. P. Smyrnélis: Sur les fonctions hyperharmoniques dórdre 2. Lecture Notes in Math. 681, Springer Verlag, (1978), 277–293.