Notes on Möbius Transformations in Several Dimensions

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L. V. Ahlfors gave formulae on hyperbolic motions in his paper [1]. We shall show the properties on Möbius transformatios in several dimensions.

1. Vectors in $\mathbf{R}^n$ will be denoted by $x=(x_1, x_2, \dots, x_n)$, the inner product by xy, and the norm by $|x|$. The reflection of x in the unit sphere S^{n-1} is denoted by $x^*=x/|x|^2$. We use the notation $\mathbf{B}$ for the unit ball in $\mathbf{R}^n$, $\mathbf{B}^*$ for its exterior. The latter should rightly include ∞, but since we shall be concerned mainly with transformations that leave $\mathbf{B}$ invariant, we need not pay attention to the compactification.

The full group M^* of hyperbolic motions and reflections is generated by the reflections in spheres or planes orthogonal to S^{n-1}. The subgroup M obtained by an even number of reflections is the group of hyperbolic motions.

We use the notation $\gamma(x)$ for its image of x under $\gamma\in M$. We write $\gamma'(x)$ for the Jacobian matrix at x and $|\gamma'(x)|$ for the linear magnification. In other words, $d\gamma(x)=\gamma'(x)dx$ and $|d\gamma(x)|=|\gamma'(x)||dx|$, the ratio being the same in all directions. Observe that $|\det \gamma'(x)|=|\gamma'(x)|^n$. The conformality implies furthermore that $|\gamma'(x)|^{-1}\gamma'(x)$ is an orthogonal matrix, and consequently ${}^t\gamma'\gamma'=|\gamma'|^2$ (${}^t\gamma'$ is the transpose, and the unit matrix denoted by 1). If 0 is a fixed point, $\gamma(0)=0$, γ' is a constant orthogonal matrix.

2. As our basic notation we shall introduce

(1) $\quad T_y(x)=(1-Q(y))A_{y^*}(x)=-A_y(x^*),$

where $A_y(x)=y+\dfrac{|y|^2-1}{|x-y|^2}(x-y)$

and $\quad Q(y)_{ij}=\dfrac{y_i y_j}{|y|^2}$.

Note that

$$Q(y)^2=Q(y) \text{ and } (1-2Q(y))^2=1.$$

For $|y|<1$, $T_y(x)$ represents a transformation in M which carries y to 0. If S has the same property, T_yS^{-1} leaves 0 fixed and is hence an orthogonal transformation. Thus the most general transformation in M (or M^*) which carries y to 0 is of the form U_yT_y with orthogonal U_y.

From (1), we obtain

$$(2) \quad T_y'(x)=\frac{1-|y|^2}{|y|^2|x-y^*|^2}(1-2Q(y))(1-2Q(x-y^*)).$$

In particular, we have

$$|T_y'(0)|=1-|y|^2 \text{ and } |T_y'(y)|=1/1-|y|^2.$$

We denote the set of elements which fix 0 by M_0. Let $\gamma\in M-M_0$. Then γ has the form U_yT_y, where $y=\gamma^{-1}(0)$ and $U_y\in SO(n)$.

Noting that $|U|=1$, we can derive

$$(3) \quad |\gamma'(x)|=|T_y'(x)|=(1-|y|^2)/|y|^2|x-y^*|^2.$$

The chain rule proves that

$$|\gamma(x)|=|\gamma(x)-\gamma(y)|=|\gamma'(x)|^{1/2}|\gamma'(y)|^{1/2}|x-y|.$$

Using (3), we have

$$|\gamma'(x)|=\frac{1-|\gamma(x)|^2}{1-|x|^2}.$$

The sphere $I(\gamma)=\{x\,|\,|\gamma'(x)|=1\}$ is called the isometric sphere of γ. We easily see that the center of $I(\gamma)$ is y^* and the radius is $(1-|y|^2)^{1/2}/|y|$. We denote U_yT_y by γ_y. Represent by $I(\gamma_a)$, $I'(\gamma_a)$, $I(\gamma_b)$, $I'(\gamma_b)$, $I(\gamma_a\gamma_b)$ the isometric spheres of γ_a, γ_a^{-1}, γ_b, γ_b^{-1}, $\gamma_a\gamma_b$ respectively; by g_a, g_a', g_b, g_b', g_{ab} their respective centers; and by r_a, r_a', r_b, r_b', r_{ab} their radii.

Proposition 1. $\dfrac{r_a r_b}{|g_b'-g_a|}=r_{ab}$.

Proof. Noting that $T_b^{-1}=T_{-b}$, we see

$$\begin{aligned} g_{ab}&=(\gamma_a\gamma_b)^{-1}(\infty)\\ &=(U_aT_aU_bT_b)^{-1}(\infty)\\ &=T_b^{-1}(U_b^{-1}(a^*))\\ &=T_{-b}(U_b^{-1}(a^*))\\ &=(1-2Q(-b))A_{(-b)}{}^*(U_b^{-1}(a^*)). \end{aligned}$$

A short computation leads to

$$(4) \quad |A_{(-b)}{}^*(U_b^{-1}(a^*))|^2=|U_b^{-1}(a^*)+b^*|^{-2}\{|b^*|^2|U_b^{-1}(a^*)|^2+1+2b^*U_b^{-1}(a^*)\}.$$

It is easily seen that

$$
\begin{aligned}
g_b' &= \gamma_b(\infty) \\
&= U_b T_b(\infty) \\
&= U_b(1-2Q(b))b^* \qquad (5)\\
&= U_b(-b^*).
\end{aligned}
$$

Together with (4), we have

$$
\begin{aligned}
r_{ab}^2 &= |g_{ab}|^2 - 1 \\
&= |A_{(-b)}{}^*(U_b^{-1}(a^*))|^2 - 1 \\
&= |U_b^{-1}(a^*) + b^*|^{-2}\{|b^*|^2|U_b^{-1}(a^*)|^2 + 1 - |U_b^{-1}(a^*)|^2 - |b^*|^2\} \\
&= |a^* - U_b(-b^*)|^{-2}(1-|a|^2)(1-|b|^2)|a|^{-2}|b|^{-2} \\
&= \frac{r_a^2\, r_b^2}{|g_b' - g_a|^2}.
\end{aligned}
$$

Our proposition is now completely proved.

Proposition 2. $\dfrac{r_a\, r_b}{|g_b - g_{ab}|} = r_{ab}$.

Proof. In Proposition 1, we replace r_a and r_b by r_a^{-1} and $r_a r_b$, respectively. The fact that $r_b = r_b^{-1}$ leads to our conclusion.

Proposition 3. Let Q_ρ be the sphere with the radius ρ at the origin 0. If the centers of all isometric spheres lie in Q_ρ, then the radius of isometric sphere is less than 2ρ.

Proof. By Propositions 1 and 2, we have

$$|g_{ab} - g_b| = \frac{r_b^2}{|g_b' - g_a|}.$$

Since the centers of isometric spheres lie in Q_ρ, we see that $|g_{ab} - g_b| < 2\rho$ and $|g_b' - g_a| < 2\rho$. Thus we have

$$(6) \quad r_b < 2\rho.$$

Proposition 4. Under the same assumption as in Proposition 3, the number of isometric spheres with radii exceeding a given positive quantity is finite.

Proof. Let $I(\gamma_a)$ and $I(\gamma_b)$ be any two different isometric spheres with radii greater than k, a positive quantity. Then $\gamma_a \gamma_b$ is not the identical transformation, and from (6),

$$|g_b' - g_a| = \frac{r_a\, r_b}{r_{ab}} > \frac{k^2}{2\rho}.$$

The distance between the centers of isometric spheres with radii exceeding k has thus a positive lower bound. Since the centers of all such spheres lie in the sphere Q_ρ, their number must be finite.

Using Propotition 3, we easily show next two propositions.

Propositions 4. The transformations of discrete subgroup of M are denumerable.

Proposition 5. Given any infinite sequence of distinct isometric spheres $I(\gamma_{a(1)})$, $I(\gamma_{a(2)})$, ... of transformations of discrete subgroup, the radii being $r_1, r_2, \dots$, then $\lim_{n\to\infty} r_n=0$.

Proposition 6. Let R and r be the radii of two spheres. Let d be the distance between the centers of these two spheres. Then an element $\gamma\in M$ (or M^*) preserves

$$K=\frac{(R^2+r^2-d^2)^2}{4R^2r^2}.$$

Proof. The transformation γ is represented by U_yT_y. Obviously U_y preserves K. Therefore we have only to show that T_y preserves K. Let a and b be the centers of two spheres, respectively. We consider the images of two spheres under T_y. We denote the radii of resulted spheres by R', r', respectively; by α, β their respective centers. We see that

$$R'^2= R^2(1-|y|^2)/(|a|^2|y|^2-R^2|y|^2-2ay+1)^2,$$
$$r'^2 = r^2(1-|y|^2)/(|b|^2|y|^2-r^2|y|^2-2by+1)^2,$$
$$\alpha = -(|a|^2y-R^2y-2(ay)y-a+|y|^2a+y)/(|a|^2|y|^2-R^2|y|^2-2ay+1), \text{ and}$$
$$\beta = -(|b|^2y-r^2y-2(by)y-b+|y|^2b+y)/(|b|^2|y|^2-r^2|y|^2-2by+1).$$

The direct computation yields our conclusion.

We define $[x_1, x_2, x_3, x_4]$ by $\dfrac{|x_1-x_3|}{|x_1-x_4|} : \dfrac{|x_2-x_4|}{|x_2-x_3|}$ and call this the cross-ratio.

Proposition 7. An element $\gamma\in M$ preserves the cross-ratio.

Proof. We have only to show that T_y preserves the cross-ratio. Noting that $|a^*-b^*|=|a-b|/|a||b|$, we obtain

$$(7) \quad |A_y^*(x_1)-A_y^*(x_2)|=\frac{||y^*|^2-1|\,|x_1-x_2|}{|x_1-y^*|\,|x_2-y^*|}.$$

From (7), we have

(8) $|T_y{}^*(x_1)-T_y{}^*(x_2)|=|(1-2Q(y))(A_y{}^*(x_1)-A_y{}^*(x_2))|$

$$=\frac{||y^*|^2-1|\,|x_1-x_2|}{|x_1-y^*|\,|x_2-y^*|}.$$

By (8), we obtain $[T_y(x_1), T_y(x_2), T_y(x_3), T_y(x_4)]=[x_1, x_2, x_3, x_4]$.

We easily see next proposition.

Proposition 8.

(i) $|(\gamma^{-1})'(\gamma(x))|=|\gamma'(x)|^{-1}$.

(ii) $|(\gamma_a\gamma_b)'(x)|=|\gamma_a'(\gamma_b(x))|\,|\gamma_b'(x)|$.

By Proposition 8, we have

Proposition 9. γ transforms Int $(I(\gamma))$ (resp. Ext $(I(\gamma))$) into Ext $(I(\gamma^{-1}))$ (resp. Int $(I(\gamma^{-1}))$) and γ^{-1} transforms Int $(I(\gamma^{-1}))$ (resp. Ext $(I(\gamma^{-1}))$ into Ext $(I(\gamma))$ (resp. Int $(I(\gamma))$). If $I(\gamma_a)$ and $I(\gamma_b')$ are exterior to one another, then $I(\gamma_a\gamma_b)$ is contained in $I(\gamma_b)$.

Let G be a discrete subgroup of M. Assume that $G\cap M_0=\{1\}$. In other words, 0 is not fixed point. Then every element in G has an isometric sphere.

We shall consider the set

$$D=(\bigcap_{\gamma\in G-\{1\}} \text{Ext}\,(I(\gamma)))\cap B.$$

It is easily seen that

$$D=\{x: |\gamma'(x)|<1 \text{ for } \gamma\in G-\{1\}\}\cap B$$
$$=\{x: |\gamma(x)|>|x| \text{ for } \gamma\in G-\{1\}\}\cap B.$$

Following the methods of Ford [2] and Lehner [3], we can prove that

(1) No two points in D are equivalent under G.

(2) Every point in $\bar{B}$ has its equivalent point in $\bar{D}$.

Properties (1) and (2) characterize D as a fundamental set. As an intersection of half-spaces it is convex in the non-euclidean sense. Therefore D is a non-euclidean polyhedron and it is referred to as the Poincaré fundamental polyhedron of G with respect to the origin.

References

[1] L. V. Ahlfors, Hyperbolic Motions, Nagoya Math. J. **29** (1967) 163-166.

[2] L. Ford, Automorphic functions, 2nd ed., Chelsea, New York, 1951.

[3] J. Lehner, Discontinuous groups and automorphic functions, Amer. Math. Soc., Mathematical Surveys, No. **8**, Providence, 1964.