# Solutions of Maxwell's Equations by means of Regular Functions of a Biquaternion Variable.

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In previous papers[1], we have developed a theory of functions of a biquaternion variable and have shown that regular functions of a biquaternion variable satisfy Maxwell's equations in a similar sense as analytic functions of a complex variable satisfy Cauchy-Riemann's equations.

Contrary to the quaternions theory, in our theory, the boundary condition and the initial condition for a regular function are fundamentally different so that we have to introduce two different types of regular functions to deal with a boundary value problem or an initial value problem.

## 1. Derivation of regular polynomial functions $P_{n_1n_2n_3}(X)$.

We introudce regular polynomial functions $P_{n_1n_2n_3}(X)$ in the real biquaternion domain through the following generating functions.

Let a generating function $F_n(X, \underline{t})$ be defined as:

$$(1)\quad F_n(X, \underline{t})=[K(X, \underline{t})^n]=[\underline{t}x_0+(\underline{t}\cdot\underline{x})]^n$$
$$=[t_1(x_1+x_0e_1)+t_2(x_2+x_0e_2)+t_3(x_3+x_0e_3)]^n$$

where $e_1$, $e_2$, $e_3$ are the unit biquaternions satisfying the same commutation relations as Pauli's matrix: $e_k^2=1$, $(k=1, 2, 3)$, $e_ke_l=-e_le_k=\underline{i}e_m$, $\underline{i}=\sqrt{-1}$, and $(k, l, m)$ is an even permutation of (1, 2, 3), $\underline{t}$ and $\underline{x}$ are defined by $\underline{t}=t_1e_1+t_2e_2+t_3e_3$, $\underline{x}=x_1e_1+x_2e_2+x_3e_3$.

Expanding $F_n(X, \underline{t})$ in power series of $t_i$ $(i=1, 2, 3)$, we find: the following definition of $P_{n_1n_2n_3}(X)$:

$$(2)\quad K^n(X, t)=\sum_{n_1+n_2+n_3=n} n!\,P_{n_1n_2n_3}(X)t_1^{n_1}t_2^{n_2}t_3^{n_3}.$$

We call the polynomials $P_{n_1n_2n_3}(X)$ as "Fueter's polynomials"[2]. We can see easily from (1) that $K^n(X, \underline{t})$ is both side $D_x$ regular:

$$(3)\quad D_xF_n(X, \underline{t})=0,\ K^n(X, \underline{t})D_x=0,$$

where $D_x=\frac{\partial}{\partial x_0}-e_1\frac{\partial}{\partial x_1}-e_2\frac{\partial}{\partial x_2}-e_3\frac{\partial}{\partial x_3}=\sum_{\mu=0}^{3}e_\mu\frac{\partial}{\partial x_\mu}$.

Proof: We see that:

$$(4)\quad D_xK^m(X,\underline{t})=\sum_{k=0}^{3}e_k\frac{\partial}{\partial x_k}K^m=\sum_{r=1}^{m}K^{m-r}\underline{t}K^{r-1}-\sum_{r=1}^{m}\sum_{i=1}^{3}e_iK^{m-r}t_iK^{r-1}$$
$$=\sum_{r=1}^{m}[K^{m-r}\underline{t}K^{r-1}-\underline{t}K^{m-1}]=0.$$

The last equality is obtained by that $K$ and $\underline{t}$ commute: $K\underline{t}=\underline{t}K$. The same is true for $K^nD_x=0$.

Since $D_xK^n=0$ and $t_k$ are independent vaiables, we find each $P_{n_1n_2n_3}(X)$ should be $D_x$ left regular:

$$(5)\quad D_x P_{n_1n_2n_3}(X)=0.$$

Similarly we have the relation: $P_{n_1n_2n_3}(X)=0$.

## 2. Exponential functions.

Let an exponential function be defined as:

$$(6)\quad e^{iK(X,\underline{t})}=\sum_{n=0}^{+\infty}\frac{(i)^n}{n!}[\underline{t}x_0+(\underline{t}\cdot\underline{x})]^n$$

Obviousely, from (3), the above exponential function is both side $D_x$ regular:

$$(7)\quad D_xe^{i[\underline{t}x_0+(\underline{t}\cdot\underline{x})]}=e^{i[\underline{t}x_0+(\underline{t}\cdot\underline{x})]}D_x=0.$$

## 3. Fourier representation of the regular functions.

From (1), we see that the exponential functions defined by (6) is a special solution of $D_x\Phi(X)=0$. A general solution of $D_x\Phi(X)=0$ can be obtained by a superposition of the special solutions exp (iK$(X,\underline{t})$ of (6) over the parameters $t_1$, $t_2$, $t_3$:

$$(8)\quad \Phi(X)=\iiint_{-\infty}^{+\infty}e^{i[\underline{t}x_0+(\underline{t}\cdot\underline{x})]}A(\underline{t})dt_1dt_2dt_3,$$

where $A(t)$ is any function of the parameters $t_1$, $t_2$, $t_3$ and not necessarily be a scalar function but is a biquaternion function satisfying a certain constraint:

$\iiint_{-\infty}^{+\infty}|N(A)|dt_1dt_2dt_3<+\infty$, where $N(A)$ is the norm of $A$: $N(A)=a_0^2-a_1^2-a_2^2-a_3^2$, $A=\sum_{\mu=0}^{3}a_\mu e_\mu$ and $|N|$ is the absolute value of $N$.

The initial condition for $\Phi(X)$ is given by putting $x_0=0$ in (8) as:

$$(9)\quad \Phi(X)|_{x_0=0}=G(\underline{x})=\iiint_{-\infty}^{+\infty}e^{i(x_1e_1+x_2e_2+x_3e_3)}A(\underline{t})dt_1dt_2dt_3$$

When $G(\underline{x})$ the initial condition for the function $\Phi(X)$ is given, $A(\underline{t})$ can be determined by Fourier integral theorem as:

$$(10)\quad A(\underline{t}) = \frac{1}{8\pi^3}\iiint_{-\infty}^{+\infty} e^{-i(\underline{\tau}\cdot\underline{t})}G(\underline{\tau})d\tau_1 d\tau_2 d\tau_3$$

Inserting the expression (10) for $A$ in (8), we find:

$$(11)\quad \Phi(X) = \frac{1}{8\pi^3}\iiint_{-\infty}^{+\infty} e^{i[\underline{t}x_0 - \underline{t}(\underline{\tau}\cdot\underline{t})]}G(\underline{\tau})dt_1 dt_2 dt_3 d\tau_1 d\tau_2 d\tau_3.$$

**An example of the initial value problem.**

Let the initial value of $F(X)$ for $x_0=0$ is given by:

$$F(X)|_{x_0=0} = G(\underline{x})\begin{cases} =\underline{g}, \text{ for } 0\leqq|x|\leqq a, \ \underline{g} \text{ is a constant vector biquaternion,} \\ =0, \text{ otherwise.}\end{cases}$$

Using the formula given by (11) and through some calculations, we find:

$$F(X{:}A) = \frac{2g}{\pi a^2}\int_0^{+\infty}\frac{dt}{t^3}(\sin at - at\cos at)\,[tx\cos tx_0 + \left(\frac{ix}{x}\right)\sin tx_0\,[\sin tx - tx\cos tx]]$$

where $x=\sqrt{x_1^2+x_2^2+x_3^2}$.

## 4. Series expansion.

For the function $F(X)$ mentioned above, when the initial value of $F(X)$ for $x_0=0$ is given as a uniformely converging Taylor series of $x_\mu$ for $|N(X)|<1$ as

$$(12)\quad F(X)|_{x_0=0} = A(x) = \sum_{n=0}^{+\infty}\ \sum_{n_1+n_2+n_3=n}^{(ni)} x_1^{n_1}x_2^{n_2}x_3^{n_3}C_{n_1n_2n_3}$$

then, $F(X)$ is uniquely determined as

$$(13)\quad F(X) = \sum_{n=0}^{+\infty}\ \sum_{n_1+n_2+n_3=n}^{(ni)} P_{n_1n_2n_3}(X)C_{n_1n_2n_3}$$

where

$$C_{n_1n_2n_3} = \frac{\partial^n F(X)}{\partial_{x1}^{n_1}\partial_{x2}^{n_2}\partial_{x3}^{n_3}}$$

is derived from (12).

## 5. Regular functions for boundary value problems.

The $P_{n_1n_2n_3}(X)$ introduced in the previous sections are suited for the initial value problem and the formulas obtained are not accessible readily to the use of boundary value problems.

We introduce the following generating functions which are an extension of $K^n(X;\,t)$ introduced by equation (1) in the first section.

Let $K_\mu$ ($\mu=0, 1, 2, 3$) be defined as

$$(14)\quad K_\mu = (XT)_0 - (Te_\mu)^+ x_\mu,$$

and $K_0$ is the same as K of section 1 and for example:

$$K_1=t_0(x_0+x_1t_1)+t_2[x_2+e_2(e_1x_1)]+t_3[x_3+e_3(e_1x_1)]$$

By the same procedure as the previous sections, we obtain similar results to those for $P_{n_1n_2n_3}(X)$.

We can show that

$$(15)\quad D_xK_\mu^n(X, T)=0.$$

Similarly, the exponential functions are derived as:

$$(16)\quad \phi(X;\, t_0,\, t_2,\, t_3)=exp[iK_1]$$

and $\phi$ satisfies $D_x\phi=0$.

A general solution of the above equation is:

$$(16)\quad \Phi(X) = \int\!\!\int\!\!\int_{-\infty}^{+\infty} exp[iK_1]A(t_0,\, t_2,\, t_3)dt_0dt_2dt_3.$$

The domian $[T_1^3]$ of the integration over $t_0$, $t_2$, $t_3$ is determined by the convergence of the integral which will be described as:

$$\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(t_2-t_3)x_1}|A(t_0,\, t_2,\, t_3)|dt_2dt_3<+\infty$$

$$\int\!\!\int\!\!\int_{-\infty}^{+\infty} |A(t_0,\, t_2,\, t_3)|dt_0<+\infty.$$

Let a boundary condition for a left $D_x$ regular function $\Phi(X)$ be given at $x_0=0$ by a Fourier integral form as:

$$\Phi(X)|_{x_1=0}=G(x_0,\, x_2,\, x_3)=\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{i(x_0t_0+x_2t_2+x_3t_3)}A(t_0,\, t_2,\, t_3)dt_0dt_2dt_3$$

then $\Phi(X)$ is given as:

$$(17)\quad \Phi(X) =\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}\!\!\cdots\!\int exp[i\{t_0(x_0-\tau_0)+t_2(x_2-\tau_2)+t_3(x_3-\tau_3)+ t_0+t_2e_2+t_3e_3\}]d\tau_0d\tau_2d\tau_3\cdot A(t_0,\, t_2,\, t_3)dt_0dt_2dt_3.$$

**A simple example of boundary condition.**

We deal with a two spatial dimension where the system does not depend on the third coordinate $x_3$:

$$\phi(x_0,\, x_2,\, x_3)=\frac{1}{2\pi}e^{ik_0x_0}\int\!\!\int_{-\infty}^{+\infty} e^{it_2(x_2-\tau_2)}e^{i(k_0+t_2e_2)e_1x_1}g(k_0,x_2)dt_0d\tau_2$$

Now we impose the boundary condition:

$$g(k_0,\, t_2)=e_2:\quad \text{for } |x_2|<a_2,$$
$$=0:\quad \text{for } |x_2|>a_2>0,$$

and the initial condition satisfies the "vector condition":

$\underline{g}(k_0, x_2)=g(k_0, x_2)e_2$ which is perpendicular to $(k_0+t_2e_2)e_1x_1$.

Putting those constraints in eq. (17), we have:

$$\phi(x_0, x_1, x_2)=\frac{e^{ik_0x_0}}{2\pi}\int_{\tau_2-\infty}^{+\infty}\int_{-2}^{a_0} e^{it_2(x_2-\tau_2)}e^{i(k_0+t_2e_2)e_1x_1}g(k_0, x_2)e_2dt_1d\tau_2$$

Putting the expression in a real function form as:

$$\phi(X)=\frac{e^{ik_0x_0}}{\pi}\begin{cases}\displaystyle\int_{-\infty}^{+\infty}\int_0^{k_0}\cos[t_2(x_2-\tau_2)][\cos f'x_1+i\alpha'e_1\sin f'x_1]- \\ \qquad\qquad \beta'e_3\sin[t_2(x_2-\tau_2)]\sin f'x_1]dt_2d\tau_2 \\ \displaystyle\int_{-\infty}^{+\infty}\int_{k_0}^{a_2}\cos[t_2(x_2-\tau_2)][\cosh fx_1+\alpha e_1\sinh fx_1]+ \\ \qquad\qquad i\beta e_3\sin[t_2(x_2-\tau_2)]\sinh fx_1dt_2d\tau_2.\end{cases}$$

where $f$, $f'$ are given below and the range of variables $t_2$ and $\tau_2$ are $t_2=0$ to $a_2$, $\tau_2=-\infty$ to $+\infty$, $\alpha=ik_0/f$, $\alpha'=ik_0/f'$, $\beta=t_2/f$, $\beta'=t_2/f'$ and $f=\pm\sqrt{-k_0^2+t_0^2}$, for $-k_0^2+t_0^2>0$, and $f'=\sqrt{k_0^2-t_2^2}$, for $k_0^2-t_0^2>0$.

## 6. More regular functions (the second kind).

We introduce regular functions some of which are regular in a domain including $|N(X)|\to+\infty$.

Let $L_n(T)$ be defined as:

$$(18)\quad L_n(T)=\Box_t(T^{n+2}),\ \Box_t=\frac{\partial^2}{\partial t_0^2}-\sum_{k=1}^{3}\frac{\partial^2}{\partial t_k^2},\ T=t_0+\underline{t}.$$

Simple calculations show:

$$(19)\quad L_n(T)=-4[(n+1)T^n+nT^{n-1}T^++\cdots\cdots+(n-r+1)T^{n-r}(T^+)^r+ +\cdots\cdots+(T^+)^n].$$

where $T^+=t_0-\underline{t}$, is the hyperconjugate of $T$.

Then, we can show that

$$\Box_x[L_n(YX)]=Y^+[D_t(\Box_tT^{n+2})]=0.$$

Therefore, $L_n$ is a polynomial of n-th degree in $x$, and is $D_x$ left regular. Thus, as described in the previous section, $L_n(YX)$ can be expanded by a polynomial of $P_{n_1n_2n_3}(X)$ as:

$$(20)\quad L_n(YX)=\sum_{(n_1,n_2,n_3)}^{n_1+n_2+n_3=n}P_{n_1n_2n_3}(X)R^*_{n_1n_2n_3}(Y)$$

where

$$R^*_{n_1n_2n_3}(Y)=\frac{\partial^n}{\partial^{n_1}x_1\partial^{n_2}x_2\partial^{n_3}x_3}L_n(XY),\ n=n_1+n_2+n_3.$$

Since $L_n(YX)D_y=0$, taking complex conjugate of the last equation and considering that $D_y^* = D_y$, we have:

$$D_y R_{n_1 n_2 n_3}(Y) = 0.$$

An explicit expression for $R_{n_1 n_2 n_3}(Y)$ as a function of $y_k$ is given by:

$$R_{n_1 n_2 n_3}(Y) = -4[(n+1)\sum_{(k_r)} e_{k_1} Y e_{k_2} \cdots e_{k_n} Y + n \sum_{(k_r)} e_{k_1} Y e_{k_2} \cdots e_{k_{n-1}} (e_{k_n} T)^+ +$$

$$+ \cdots\cdots + \sum_{(k_r)} e_{k_1} Y \cdots e_{k_r} Y e_{k_{r+1}} Y (e_{k_{r+1}} Y \cdots Y)^+ + \cdots\cdots$$

$$+ \sum_{(k_r)} (e_{k_1} Y e_{k_2} \cdots e_{k_n} Y)^+]$$

Now, define

$$Q_{n_1 n_2 n_3}(X) = X^{-1} N(X)^{-1} R_{n_1 n_2 n_3}(x^{-1}),$$

we have

$$(21) \quad \square_x[(YX)^{n+2} Y] = \sum_{(n_i)}^{n_1+n_2+n_3=n} P_{n_1 n_2 n_3}(X) Q^*_{n_1 n_2 n_3}(Y^{-1})$$

Then, $Q_{n_1 n_2 n_3}(X)$ is the Fueter's polynimial of the second kind[2].

Define $N_{n_1 n_2 n_3}(X^{-1}) = XN(X) P_{n_1 n_2 n_3}(X)$, we have the following equation:

$$\square_x[(YX)^{n+2}] Y = \sum_{(n_i)} R_{n_1 n_2 n_3}(X) N^*_{n_1 n_2 n_3}(Y^{-1}).$$

Comparing this equation with (21), we obtain the relation between Q's and N*'s as follows:

$$Q_{n_1 n_2 n_3}(Y) = \sum_{(r_i)} N_{r_1 r_2 r_3}(Y) a^*[n_1 n_2 n_3, r_1 r_2 r_3],$$

where $a^*$ is the complex conjugate of a: which are given by eq (22).

$$(22) \quad a[n_1 n_2 n_3; r_1 r_2 r_3] = \frac{\partial^n}{\partial y_1^{r_1} \partial y_2^{r_2} \partial y_3^{r_3}} \left[ \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} K_n(YX) \right]$$

Using those regular polynimal functions $P$, $Q$, $R$, $N$, we can expand $\square_x[(Y-X)^{-1}]$ as a series of those polynomial functions.

[1] For $|N(X)| < |N(Y)|$.

We have in this case:

$$\square_x[(Y-X)^{-1}] = \sum_{n=0}^{\infty} Y^{-1} \square_x[(XY^{-1})^{n+2}]$$

$$= \sum_{n=0}^{\infty} \sum_{(n_i)}^{n_1+n_2+n_3=n} P_{n_1 n_2 n_3}(X) Q^*_{n_1 n_2 n_3}(Y)$$

Taking complex conjugate and using (13), Q's can be obtained as:

$$Q^*_{n_1 n_2 n_3}(Y) = \left[ \frac{\partial^n [\square_x (Y-X)^{-1}]}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \right] (Y^{-1} [\square_x (XY^{-1})^{n+2}]).$$

[2] For $|N(X)| > |N(Y)|$.

$$\square_x[(Y-X)^{-1}] = \frac{-4(Y-X)^+}{[N(Y-X)]^2}$$

$$= \sum_{n=0}^{\infty} \sum_{(n_i)}^{n_1+n_2+n_3=n} (-1) N_{n_1 n_2 n_3}(X) R^*_{n_1 n_2 n_3}(Y)$$

where

$$N_{n_1n_2n_3}(X)=X^{-1}[N(X)]^{-1}P_{n_1n_2n_3}(X^{-1})$$

$$R_{n_1n_2n_3}(X)=X^{-1}[N(X)]^{-1}Q_{n_1n_2n_3}(X^{-1}).$$

The details of the derivation of the above relations will be given in Ref. (3).

## 7. Extended regular polynomial functions.

The polynomial functions discussed in previous sections are suitable for dealing with an initial value problem but are not convenient to find solutions for a given boundary condition. To overcome with the difficulty, we extend the theory to include polynomial functions which are regular and are able to deal with the boundary value problem.

We rewrite the $K_\mu$ function defined by (14) as follows:

(23) $$K_\mu(X, T)=\sum_{\mu=0}^{3} t_{\mu+\nu}[x_{\mu+\nu}-(x_\mu e_\mu)e_{\mu+\nu}],$$

where $t_{\mu+\nu}\equiv t_\mu$, $x_{\mu+\nu}=x_\rho$, $e_{\mu+\nu}=e_\rho$ where $\mu+\nu\equiv\rho$ (mod. 4), ($\rho=0$, 1, 2, 3).

Then,

(24) $$K_\mu{}^n(X, T)=\sum_{(n_r)}^{n_0+n_1+n_2+n_3=n} n!\, P^{(\mu)}_{n_0n_1n_2n_3}(X)t_0{}^{n_0}t_1{}^{n_1}t_2{}^{n_2}t_3{}^{n_3}$$

and

(25) $$P^{(\mu)}_{n_0n_1n_2n_3}(X)=\frac{1}{n!}\sum_{(n_r)}^{n_0+n_1+n_2+n_3}(x_{a_1}-\varepsilon^\mu_{a_1}x)\cdots\cdots(x_{a_n}-\varepsilon^\mu_{a_n}x)$$

where

(26) $$(\varepsilon_\mu{}^\nu)=\begin{pmatrix}1 & e_1 & e_2 & e_3\\ e_1 & 1 & -ie_3 & +ie_2\\ e_2 & +ie_3 & 1 & -ie_1\\ e_3 & -ie_2 & +ie_1 & 1\end{pmatrix}=(e_\mu e_\nu)$$

We can see from (25) and (26) that $P^{(\mu)}_{n_0n_1n_2n_3}(X)=0$ for $n_\mu\neq0$. From equation (24) that $P^{(\mu)}_{n_0n_1n_2n_3}(X)$ are left $D$ regular:

(27) $$DP^{(\mu)}_{n_0n_1n_2n_3}(X)=0, \text{ and } P^{(\mu)*}_{n_0n_1n_2n_3}(X)D=0.$$

From the definition (25), we find for $P$:

$$P^{(\mu)}_{n_0n_1n_2n_3}(X)\Big|_{x_\mu=0}=\frac{n_0!\,n_1!\,n_2!\,n_3!}{n!}x_0{}^{n_0}x_1{}^{n_1}x_2{}^{n_2}x_3{}^{n_3}\delta_{n_\mu}$$

From equations (23) and (24), we find:

$$\frac{\partial}{\partial x_0}P^{(\mu)}_{n_0n_1n_2n_3}(X)=P^{(\mu)}_{n_0-1,\,n_1n_2n_3}(X),\ (n_\mu=0),$$

$$\frac{\partial}{\partial x_1}P^{(\mu)}_{n_0n_1n_2n_3}(X)=P^{(\mu)}_{n_0n_1-1,\,n_2n_3}(X) \text{ and so on for } n_2 \text{ and } n_3.$$

Also, we get from (28) that

$$\frac{\partial^n}{\partial x_0{}^{n_0}\partial x_1{}^{n_1}\partial x_2{}^{n_2}\partial x_3{}^{n_3}} P^{(\mu)}_{n_0 n_1 n_2 n_3}(X) = \delta_{r_0 n_0}\delta_{r_1 n_1}\delta_{r_2 n_2}\delta_{r_3 n_3},$$

where $r_0+r_1+r_2+r_3=n_0+n_1+n_2+n_3$.

We can easily extend the formulas valid for $P_{n_1 n_2 n_3}(X)$ to $P^{(\mu)}_{n_0 n_1 n_2 n_3}(X)$.

In a similar manner as that derived the series expansion described in 4., we can prove the following theorem.

**Theorem.** Let a D left regular function of n-th degree polynomial F($X$) which satisfies the following boundary condition at $x_1=0$ given by a Taylor expansion as:

$$F(X)|_{x_1=0} = G(X) = \sum_{(n_0,\, n_2,\, n_3)}^{n_0+n_2+n_3=n} \frac{n!}{n_0! n_2! n_3!} C_{n_0 n_2 n_3} x_0{}^{n_0} x_2{}^{n_2} x_3{}^{n_3},$$

can be expressed uniquely by a polynomial of $P^{(1)}_{n_0 n_2 n_3}(X)$ as

$$F(X) = \sum_{(n_0,\, n_2,\, n_3)}^{n_0+n_2+n_3=n} P^{(1)}_{n_0 n_2 n_3}(X) C_{n_0 n_2 n_3},$$

where

$$C_{n_0 n_2 n_3} = \left[ \frac{\partial^n G(X)}{\partial x_0{}^{n_0}\, \partial x_2{}^{n_2}\, \partial x_3{}^{n_3}} \right], \text{ and } n=n_0+n_2+n_3.$$

The summation is taken over all different partitions of $n$ into three integars $n_0$, $n_2$, $n_3$ : $0 \leqq n_i \leqq n$.

## 8. Derivation of the solutions of Maxwell's equations from regular functions.

In order that a regular function of a biquaternion variable which we have derived in the previous sections should satisfy Maxwell's equations, the function should satisfy an initial condition.

Let the regular function be $F(X)$, then the "vector condition" : $F(X)=-F^+(X)$, is expressed as

$$F(X)|_{x_0=0} = G(\underline{x}) = -F^+(X)|_{x_0=0} = \underline{g}(\underline{x}).$$

Thus, when an initial condition for the solution of Maxwell's equations is given, the initial value of the solution is given as a vector function, we can express the solution satisfying the initial condition by equation (11).

### References

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2) R. Fueter, Comm. Math. Helv., **7**, 307 (1934-35), **8**, 371 (1936-37), **9**, 320 (1936-37), **10**, 327 (1937-38).

3) A detailed account of the theory in a book form will be available whortly: "The Quaternionic formulation of Classical Electrodynaics".