# A Remark on Star Convergences of $r$-Convergences in Ranked Spaces


Masato HIKIDA

*Department of Applied Mathematics, Okayama University of Science*
*Ridai-cho 1-1, Okayama 700, Japan*



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In this paper we use the same terminology and notations as in [1].

Let $R$ be a ranked space of indicator $\omega_0$. According to the Theorem of [1], for a point $p \in R$, the following two statements are equivalent as long as $R$ has some property:

(S) the $r$-convergence to $p$ is the star convergence;

(D) $\boldsymbol{F}(p)$ is directed (with respect to the order relation $<$).

Then does one of implications (S) $\rightarrow$ (D) and (D) $\rightarrow$ (S) hold in general? At a glance it seems that this question is affirmative. For instance, for spaces appeared so far to be examples in papers relating to ranked spaces, at least one of the implications holds even if the Theorem of [1] cannot be applied. However, the examples given below shows that the above question is negative.

Let $E=\{(x, y): x$ and $y$ are real numbers$\}$; and, for a point $p=(a, b) \in E$ and for real numbers $\varepsilon, \theta$ with $\varepsilon>0$ and $\theta \in [0, 2\pi)$, let $N(p; \varepsilon, \theta)=\{(a+\delta\cos\theta, b+\delta\sin\theta): 0<\delta\leq\varepsilon\}$ and $B(\varepsilon)=\{(x, y): x^2+y^2<\varepsilon\}$.

**Example 1** ((S) $\rightarrow$ (D) is false in general). For each $p \in E$, define preneighborhoods of $p$, $v_{\varepsilon,\theta}(p)$ ($\varepsilon>0$, $\theta\in[0, 2\pi)$), by

$$v_{\varepsilon,\theta}(p)=\begin{cases} B(\varepsilon)-N(p; \varepsilon, \theta) & \text{if } p=(0, 0) \\ N(p; \varepsilon, \theta) \cup \{p\} & \text{if } p\neq(0, 0); \end{cases}$$

and call $v_{\varepsilon,\theta}(p)$ preneighborhoods of rank $n$ if $[1/\varepsilon]=n$. Then $E$ becomes a ranked space of indicator $\omega_0$: We denote this space by $E_1$. In $E_1$, for the point $p_0\equiv(0, 0)$, $\boldsymbol{F}(p_0)$ is clearly not directed, but (S) is true. In fact, let $\{q_i\}$ be a sequence of points in $E_1$ and assume that any subsequence of $\{q_i\}$ contains a subsequence which is $r$-convergent to $p_0$. It is easy to see that, for any $\varepsilon>0$, there is an integer $i_\varepsilon\geq 0$

such that $q_i \in B(\varepsilon)$ for all $i \geq i_\varepsilon$; hence in particular, $q_i \in B(1)$ for all $i \geq i_1$. Now, since the interval $[0, 2\pi)$ is uncountable and since the family $\{N(p_0; 1, \theta)\}$ $(\theta \in [0, 2\pi))$ is disjoint, there is some $\theta^* \in [0, 2\pi)$ such that $q_i \notin N(p_0; 1, \theta^*)$ for all $i \geq i_1$. Hence, taking the $p_0$-f.s. $V \equiv \{v_{1/(i+1), \theta^*}(p_0)\}$, we have $q_i \underset{r}{\to} p_0$ $(V)$. (On the other hand, for a point $p \neq p_0$ in $E_1$, though the Theorem of [1] cannot be applied, (S) $\to$ (D) is true.)

The above argument is also valid if we replace $v_{\varepsilon,\theta}(p_0)$ $(\varepsilon > 0, \theta \in [0, 2\pi))$ by $v_{\varepsilon,\Theta}(p_0) \equiv B(\varepsilon) - \bigcup_{\theta \in \Theta} N(p_0; \varepsilon, \theta)$ $(\varepsilon > 0; \Theta$ is any non-empty proper subset of $[0, 2\pi))$.

**Example 2** ((D) $\to$ (S) is false in general). Let $\mathfrak{S} = \{\Theta : \Theta$ is a non-empty subset of $[0, 2\pi)$ which is nowhere dense in $[0, 2\pi\}$, and let $\mathfrak{F} = \{\Theta : \Theta \in \mathfrak{S}$ and $\Theta$ is finite$\}$. For each $p \in E$, define preneighborhoods of $p$, $v_{\varepsilon,\Theta}(p)$ $(\varepsilon > 0; \Theta \in \mathfrak{S}$ if $p = (0, 0)$, and $\Theta \in \mathfrak{F}$ if $p \neq (0, 0))$, by $v_{\varepsilon,\Theta}(p) = (\bigcup_{\theta \in \Theta} N(p; \varepsilon, \theta)) \cup \{p\}$; and call $v_{\varepsilon,\Theta}(p)$ preneighborhoods of rank $n$ if $[1/\varepsilon] = n$. Then $E$ becomes a ranked space of indicator $\omega_0$: We denote this space by $E_2$. In $E_2$, for the point $p_0 \equiv (0, 0)$, $\boldsymbol{F}(p_0)$ is clearly directed, but (S) is false. In fact, let $\{s_k\}$ be the sequence of points in $E_2$ defined by $s_k = (\delta_i \cos\theta_{i,j}, \delta_i \sin\theta_{i,j})$, where $i = 0, 1, 2, \cdots, j = 0, 1, \cdots, 2^{i+2} - 1$, $k = 4(2^i - 1) + j$, $\delta_i = 1/(i+1)$ and $\theta_{i,j} = 2^{-i-2} j\pi$. It is obvious that, for any $\varepsilon > 0$, there is an integer $k_\varepsilon \geq 0$ such that $s_k \in B(\varepsilon)$ for all $k \geq k_\varepsilon$. Now let $\{t_n\}$ be any subsequence of $\{s_k\}$. Then, for each $t_n$, there exists a unique $\theta_n \in [0, 2\pi)$ such that $t_n \in N(p_0; 1, \theta_n)$. It is easy to choose a subsequence $\{\theta_{n(m)}\}$ of $\{\theta_n\}$ so that the set $\Theta^* \equiv \{\theta_{n(m)} : m = 0, 1, 2, \cdots\}$ belongs to $\mathfrak{S}$. Hence, taking the $p_0$-f.s. $U \equiv \{v_{1/(m+1), \Theta^*}(p_0)\}$, we have $t_{n(m)} \underset{r}{\to} p_0$ $(U)$. Thus $\{t_n\}$ contains a subsequence which is $r$-convergent to $p_0$. But $\{s_k\}$ is not $r$-convergent to $p_0$, since, for some $p_0$-f.s. $W = \{v_{\varepsilon_k, \Theta_k}(p_0)\}$, $s_k \underset{r}{\to} p_0$ $(W)$ must imply that, for any $k$, $\Theta_k$ is everywhere dense in $[0, 2\pi)$. (On the other hand, for a point $p \neq p_0$ in $E_2$, though the Theorem of [1] cannot be applied, (D) $\to$ (S) is true.)

## Reference

[1] M. Hikida: On star convergences of $r$-convergences in ranked spaces, The Bulletin of the Okayama University of Science, **15** (1979), 1-4.