On the Bi-harmonic Green's Function of a Bi-harmonic Space

Hidematu TANAKA

Department of Fundamental Natural Science, Okayama University of Science, Ridai-cho, Okayama 700, Japan

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Introduction

Let X be a connected, locally connected and locally compact Hausdorff space with a countable basis and (X, **H**) be an elliptic bi-harmonic space in the sense of Smyrnélis [6].

We denote by (X, $\mathbf{H}_j$) (j=1, 2) the Brelot's harmonic space associated with (X, **H**) and suppose that (X, $\mathbf{H}_j$) (j=1, 2) satisfies the local proportionality axiom and (X, $\mathbf{H}_2$) is strong. Then there exists the $\mathbf{H}_2$-Green's function $G^{(2)}(x, y)$ on X.

A function $N(x, y): X\times X\rightarrow(0, +\infty]$ is called the bi-harmonic Green's function of (X, **H**) associatəd with $G^{(2)}(x, y)$ if for each $y\in X$, the couple $(N(\cdot, y), G^{(2)}(\cdot, y))$ is an **H**-potential on X and **H**-harmonic on X-$\{y\}$. We shall show that the bi-harmonic Green's function of (X, **H**) associated with $G^{(2)}(x, y)$ exists if and only if there exists the $\mathbf{H}_1$-Green's function $G^{(1)}(x, y)$ on X and, for each $y\in X$,

$$\int G^{(1)}(x, z)G^{(2)}(z, y)\, d\alpha(z) < +\infty$$

for some $x\in X$, where α is the composing measure associated with $G^{(1)}(x, y)$. In this case we have

$$N(x, y) = \int G^{(1)}(x, z)G^{(2)}(z, y)\, d\alpha(z).$$

We shall give the integral representation of an **H**-potential on X by using this couple $(N(\cdot, y), G^{(2)}(\cdot, y))$ of kernels.

1. Bi-harmonic spaces

For an open set $U\neq\emptyset$ in X, we denote by **C**(U) the real vector space of finite continuous functions on U. An element (h_1, h_2) in $\mathbf{C}(U)\times\mathbf{C}(U)$ is called compatible if $h_1=0$ on an open subset U′ of U implies $h_2=0$ on U′. Let **H** be an application $U\rightarrow\mathbf{H}(U)$, where $\mathbf{H}(U)$ is a real vector subspace of compatible couples in $\mathbf{C}(U)\times\mathbf{C}(U)$.

A relatively compact open set ω is called **H**-regular if for any couple (f_1, f_2) of finite continuous functions on the boundary $\partial\omega$ of ω, there exists a unique $(h_1, h_2)\in \mathbf{H}(\omega)$ such that:

(i) $\lim_{x\to a} h_j(x)=f_j(a)$ for any $a\in\partial\omega$ (j=1, 2);

(ii) $f_j\geqq 0$ (j=1, 2) implies $h_1\geqq 0$ and $f_2\geqq 0$ implies $h_2\geqq 0$.

For an **H**-regular set ω, there exists a unique system $(\mu^\omega{}_x, \nu^\omega{}_x, \lambda^\omega{}_x)$ of positive Radon measures on $\partial\omega$ such that

$$h_1(x) = \int f_1 d\mu^\omega{}_x + \int f_2 d\nu^\omega{}_x,\ h_2(x) = \int f_2 d\lambda^\omega{}_x.$$

We say that (X, **H**) is an elliptic bi-harmonic space in the sense of Smyrnélis [6] if it satisfies the following four axioms.

Axiom I. **H** is a sheaf on X.

For an open set U in X, an element in **H**(U) is called **H**-harmonic on U.

Axiom II. The **H**-regular open sets form a basis of X.

Let U be an open set in X. A couple (v_1, v_2) of functions on U is called **H**-hyperharmonic on U if

(i) v_j is lower semi-continuous and $>-\infty$ on U (j=1, 2),

(ii) $v_1(x)\geqq\int v_1 d\mu^\omega{}_x+\int v_2 d\nu^\omega{}_x$ and $v_2(x)\geqq\int v_2 d\lambda^\omega{}_x$ for any **H**-regular neighborhood ω of x with $\bar{\omega}\subset$U.

An **H**-hyperharmonic couple (s_1, s_2) on U is called **H**-superharmonic on U if s_j is not identically $+\infty$ on any connected component of U (j=1, 2) and an **H**-superharmonic couple (p_1, p_2) on X is called an **H**-potential on X if $p_j\geqq 0$ and, for any $(h_1, h_2)\in\mathbf{H}$(X), $h_j=0$ so far as $0\leqq h_j\leqq p_j$ (j=1, 2). The set of all **H**-superharmonic couples (resp. **H**-potentials) on X is denoted by **S**(X) (resp. **P**(X)). For an open set U, denote by **H***(U) the set of all **H**-hyperharmonic couples on U and put $\mathbf{H}^*{}_1(\mathrm{U})=\{v_1: (v_1, 0)\in\mathbf{H}^*(\mathrm{U})\}$, $\mathbf{H}^*{}_2(\mathrm{U})=\{v_2: (v_1, v_2)\in\mathbf{H}^*(\mathrm{U})$ for some $v_1\}$ and $\mathbf{H}_j(\mathrm{U})=\mathbf{H}^*{}_j(\mathrm{U})\cap[-\mathbf{H}^*{}_j(\mathrm{U})]$ (j=1, 2)

Axiom III. (i) $\mathbf{H}^*{}_j$(X) separates the points of X linearly (j=1, 2).

(ii) On each relatively compact open set U there exists a strictly positive $h_j\in\mathbf{H}_j$(U) (j=1, 2).

Axiom IV. If U is a domain in X and $\{h_j{}^{(n)}\}_n$ is an increasing sequence of functions in $\mathbf{H}_j$(U), then either $\sup_n h_j{}^{(n)}=+\infty$ or $\sup_n h_j{}^{(n)}\in\mathbf{H}_j$(U) (j=1, 2).

Set $\mathbf{H}_j=\{\mathbf{H}_j(\mathrm{U})\}_{\mathrm{U}:\text{open}}$. It is shown by Theorem 1.29 in [6] that (X, $\mathbf{H}_j$) (j=1, 2) is a Brelot's harmonic space. We call (X, $\mathbf{H}_j$) (j=1, 2) the Brelot's harmonic space associated with (X, **H**). The set of all $\mathbf{H}_j$-superharmonic functions (resp.

$\mathbf{H}_j$-potentials) on X is denoted by $\mathbf{S}_j(X)$ (resp. $\mathbf{P}_j(X)$)($j=1, 2$).

2. Some lemmas

Let (X, **H**) be an elliptic bi-harmonic space and (X, $\mathbf{H}_j$) ($j=1, 2$) be the Brelot's harmonic space associated with (X, **H**).

For a real-valued function f on an open set $U\subset X$, we set

$$\Gamma f(x)=\limsup_{\substack{\omega\searrow x\\ U\supset\omega\supset\omega\\ \omega:\ \mathbf{H}\text{-regural}}}\frac{f(x)-\int f d\mu^{\omega}{}_x}{\int d\nu^{\omega}{}_x},$$

$$\Gamma' f(x)=\liminf_{\substack{\omega\searrow x\\ U\supset\omega\supset\omega\\ \omega:\ \mathbf{H}\text{-regular}}}\frac{f(x)-\int f d\mu^{\omega}{}_x}{\int d\nu^{\omega}{}_x}.$$

By Theorem 11.3 in [7] we have

Lemma 1. *Let (v_1, v_2) be a couple of lower semi-continuous functions on an open set U. Then the following three conditions are equivalent:*

(i) $(v_1, v_2)\in\mathbf{H}^*(U)$;

(ii) $v_2\in\mathbf{H}^*{}_2(U)$ *and* $\Gamma' v_1(x)\geqq v_2(x)$ *for any* $x\in U$ *with* $v_1(x)<+\infty$;

(iii) $v_2\in\mathbf{H}^*{}_2(U)$ *and* $\Gamma v_1(x)\geqq v_2(x)$ *for any* $x\in U$ *with* $v_1(x)<+\infty$.

We say that (X, $\mathbf{H}_1$) satisfies the local proportionality axiom if the totality of $\mathbf{H}_1$-regular sets U satisfying the condition (*) forms a basis of X:

(*) Any two $\mathbf{H}_1$-potentials on U with common one-point support are proportional.

If (X, $\mathbf{H}_1$) satisfies the local proportionality axiom, then there exists a consistent system $(G_{\omega}{}^{(1)})_{\omega:\ \mathbf{H}\text{-regular}}$ of $\mathbf{H}_1$-Green's functions (see Theorem 1.6 in [11] and § 4 in [10]). That is, to each **H**-regular set ω there corresponds a function $G_{\omega}{}^{(1)}(x, y)$ on $\omega\times\omega$ having the following properties:

(i) for each $y\in\omega$, $G_{\omega}{}^{(1)}(\cdot, y)$ is an $\mathbf{H}_1$-potential on ω and $\mathbf{H}_1$-harmonic on $\omega-\{y\}$;

(ii) if ω' is an **H**-regular set with $\omega'\subset\omega$ and $y\in\omega'$, then the function $G_{\omega}{}^{(1)}(x, y)-G_{\omega'}{}^{(1)}(x, y)$ of x is $\mathbf{H}_1$-harmonic on ω';

(iii) for each $\mathbf{H}_1$-potential p on ω there exists a unique positive Radon measure m on ω such that

$$p(x)=\int G\omega^{(1)}(x, y)dm(y).$$

By Theorem 9 in [10] we have

Lemma 2. *Suppose that* $(X, \mathbf{H}_1)$ *satisfies the local proportionality axiom. For a consistent system* $(G_\omega{}^{(1)})_{\omega:\mathbf{H}\text{-regular}}$ *of* $\mathbf{H}_1$*-Green's functions, there exists a unique positive Rabon measure* α *on* X *such that*

$$\nu^\omega{}_x=\int G_\omega{}^{(1)}(x, y)\lambda^\omega{}_y\, d\alpha(y)$$

for any **H**-*regular set* ω *and any* $x\in\omega$ (*i. e.*,

$$\int fd\,\nu^\omega{}_x=\int G_\omega{}^{(1)}(x, y)\left(\int fd\lambda^\omega{}_y\right)d\alpha(y)$$

for any finite continuous function f *on* X).

This positive Radon measure α is called a composing measure of (X, **H**) associated with $(G_\omega{}^{(1)})_{\omega:\mathbf{H}\text{-regular}}$. By virtue of the compatibility of bi-harmonic couples, this measure α is everywhere dense in X. From now on we suppose that $(X, \mathbf{H}_1)$ satisfies the local proportionality axiom and α is the composing measure of (X, **H**) associated with $(G_\omega{}^{(1)})_{\omega:\mathbf{H}\text{-regular}}$.

Lemma 3. *Let* ω *be an* **H**-*regular set and* f *be a bounded Borel measurable function on* ω. *Then the function*

$$\int G_\omega{}^{(1)}(\cdot, y)f(y)d\alpha(y)$$

is continuous on ω *and equal to 0 on* $\partial\omega$.

Proof. We may assume that $0\leqq f\leqq 1$. Let U be a relatively compact open set with $U\supset\bar{\omega}\supset\omega$. By Axiom III (ii), there exists a strictly positive $h_2\in\mathbf{H}_2(U)$. The couple $\left(\int G_\omega{}^{(1)}(\cdot, y)h_2(y)d\alpha(y), h_2\right)$ being in $\mathbf{H}(\omega)$, $\int G_\omega{}^{(1)}(\cdot, y)h_2(y)\, d\alpha(y)$ is a continuous $\mathbf{H}_1$-potential on and equal to *0* on $\partial\omega$. Since $f(y)\leqq Mh_2(y)\,(\forall y\in\omega)$ for some constant $M>0$,

$$\int G_\omega{}^{(1)}(\cdot, y)f(y)d\alpha(y)\leqq M\cdot\int G_\omega{}^{(1)}(\cdot, y)h_2(y)d\alpha(y).$$

Therefore $\int G_\omega{}^{(1)}(\cdot, y)f(y)d\alpha(y)$ is an $\mathbf{H}_1$-potential on ω and equal to 0 on $\partial\omega$. On the otherh and,

$$\int G_\omega{}^{(1)}(\cdot, y)f(y)d\alpha(y)$$
$$=M\int G_\omega{}^{(1)}(\cdot, y)h_2(y)\,d\alpha(y)-\int G_\omega{}^{(1)}(\cdot, y)\left(M-\frac{f(y)}{h_2(y)}\right)h_2(y)d\alpha(y).$$

Hence $\int G_\omega^{(1)}(\cdot, y)f(y)d\alpha(y)$ is upper semi-continuous and so it is continuous on ω.

Let v_2 be in $\mathbf{H}_2^*(X)$ and $v_2 \geqq 0$. We set

$$\mathbf{E}_{v_2} = \{w_1 : w_1 \geqq 0 \text{ and } (w_1, v_2) \in \mathbf{H}^*(X)\}$$

and $v_1 = \inf \mathbf{E}_{v_2}$. Then $v_1 \in \mathbf{E}_{v_2}$ by Lemma 11.6 in [7]. We call this function v_1 the pure hyperharmouic function of order 2 associated with v_2 (see [7] and [9]). We have the following lemma. This proof is similar to that of Lemma 11.8 in [7]. We shall note that the proof is given without the assumption of strongness of $(X, \mathbf{H})$.

Lemma 4. *Let v_2 be in $\mathbf{H}^*_2(X)$ with $v_2 \geqq 0$ and v_1 be the pure hyperharmonic function of order 2 associated with v_2. Then*

$$v_1(x) = \int v_1 d\mu^\omega{}_x + \int G_\omega^{(1)}(x, y)v_2(y)d\alpha(y)$$

for any $\mathbf{H}$-regular set ω and any $x \in \omega$.

Proof. Let ω be an $\mathbf{H}$-regular set. We put

$$w_1(x) = \begin{cases} \inf \left(v_1(x), \int v_1 d\mu^\omega{}_x + \int G_\omega^{(1)}(x, y)v_2(y)d\alpha(y)\right) & \text{for } x \in \omega \\ v_1(x) & \text{for } x \in C\omega \end{cases}$$

and $w_2(x) = v_2(x)$ on X. It suffices to show $(w_1, w_2) \in \mathbf{H}^*(X)$. Since v_j(j=1, 2) is non-negative lower semi-continuous, there exists an increasing sequence $\{f^{(n)}{}_j\}_n$ (j=1, 2) of non-negative continuous functions on X such that

$$\lim_{n\to\infty} f_j^{(n)}(x) = v_j(x) \quad (j=1, 2).$$

Set

$$r_1^{(n)}(x) = \int f_1^{(n)} d\mu^\omega{}_x + \int G_\omega^{(1)}(x, y)f_2^{(n)}(y)d\alpha(y)$$

for $x \in \omega$. Then, for any $x \in \omega$,

$$\begin{aligned} \Gamma r_1^{(n)}(x) &= \limsup_{\substack{\omega' \searrow x \\ \omega \supset \bar{\omega}' \supset \omega' \\ \omega' : \mathbf{H}\text{-regular}}} \frac{r_1^{(n)}(x) - \int r_1^{(n)} d\mu^{\omega'}{}_x}{\int d\nu^{\omega'}{}_x} \\ &= \limsup_{\omega' \searrow x} \frac{\int G_{\omega'}^{(1)}(x, y)f_2^{(n)}(y)d\alpha(y)}{\int G_{\omega'}^{(1)}(x, y)\left(\int d\lambda^{\omega'}{}_y\right)d\alpha(y)} \\ &= \limsup_{\omega' \searrow x} \frac{\int G^{(1)}{}_{\omega'}(x, y)f_2^{(n)}(y)d\alpha(y)}{\int G_{\omega'}^{(1)}(x, y)d\alpha(y)} \cdot \frac{\int G_{\omega'}^{(1)}(x, y)d\alpha(y)}{\int G_{\omega'}^{(1)}(x, y)\left(\int d\lambda^{\omega'}{}_y\right)d\alpha(y)} \\ &= f_2^{(n)}(x). \end{aligned}$$

Since $\Gamma v_1 \geqq v_2$, $\Gamma(v_1 - r_1^{(n)})(x) \geqq v_2(x) - f_2^{(n)}(x) \geqq 0$ on ω.
Hence $v_1 - r_1^{(n)} \in \mathbf{H}^*(\omega)$. On the other hand,

$$(v_1(x) - r_1^{(n)}(x)) + \int G_\omega^{(1)}(x, y) f_2^{(n)}(y) d\alpha(y)$$
$$= v_1(x) - \int f_2^{(n)} d\mu^\omega_x \geqq 0$$

and $\int G_\omega^{(1)}(\cdot, y) f_1^{(n)}(y) d\alpha(y)$ being an $\mathbf{H}_1$-potential on ω, we have $v_1(x) \geqq r_1^{(n)}(x)$ on ω. Letting $n \to \infty$, we have

$$v(x) \geqq \int v_1 d\mu^\omega_x + \int G_\omega^{(1)}(x, y) v_2(y) d\alpha(y)$$

on ω. Since $(w_1, w_2)|_\omega \in \mathbf{H}^*(\omega)$ and

$$\liminf_{\omega \ni x \to z} w_1(x) \geqq v_1(z) \quad (\forall z \in \partial\omega),$$

$(w_1, w_2) \in \mathbf{H}^*(X)$ by Proposition 1.21 in [6]. This compltetes the proof.

If there exists the $\mathbf{H}_1$-Green's function $G^{(1)}(x, y)$ on X, then we shall define a consistent system $(G_\omega^{(1)})_{\omega:\,\mathbf{H}\text{-regular}}$ of $\mathbf{H}_1$-Green's function by setting

$$G_\omega^{(1)}(x, y) = G^{(1)}(x, y) - \int G^{(1)}(z, y) d\mu^\omega_x(z).$$

The composing measure associated with this system is called the composing measure of $(X, \mathbf{H})$ associated with $G^{(1)}(x, y)$.

Lemma 5. *Let s_2 be in $\mathbf{S}_2(X)$ with $s_2 \geqq 0$ and $s_2 \neq 0$. Then the following conditions are equivalent:*

(i) *there exisis $s_1 \geqq 0$ such that $(s_1, s_2) \in \mathbf{S}(X)$;*

(ii) *there exists the $\mathbf{H}_1$-Green's function $G^{(1)}(x, y)$ on X and for some $x \in X$,*

$$\int G^{(1)}(x, y) s_2(y) d\alpha(y) < +\infty,$$

where α is the composing measure of $(X, \mathbf{H})$ associated with $G^{(1)}(x, y)$.

Proof. (i)$\Rightarrow$(ii). Since there exists $s_1 \geqq 0$ such that $(s_1, s_2) \in \mathbf{S}(X)$, the pure hyperharmonic function p_1 of order 2 associated with s_2 is a strictly positive $\mathbf{H}_1$-potential on X (see Remark c) in [9]). Hence by Theorem 18.1 in [4] there exists the $\mathbf{H}_1$-Green's function $G^{(1)}(x, y)$ on X. Let α be the composing measure of $(X, \mathbf{H})$ associated with $G^{(1)}(x, y)$. Since $p_1 \in \mathbf{P}_1(X)$, there exists a unique positive Radon measure m on X such that

$$p_1(x) = \int G^{(1)}(x, y) dm(y).$$

By Lemma 4 we have

$$\int G_\omega^{(1)}(x, y) dm(y) = \int G_\omega^{(1)}(x, y) s_2(y) d\alpha(y)$$

for any $\mathbf{H}$-regular set ω and any $x \in \omega$. Hence $m = s_2\alpha$ on ω for any $\mathbf{H}$-regular set

ω. Therefore

$$p_1=\int G^{(1)}(\cdot\,,y)s_2(y)d\alpha(y)$$

and so for some $x\in X$,

$$\int G^{(1)}(x,y)s_2(y)d\alpha(y)<+\infty\,.$$

(ii)$\Rightarrow$(i). Put

$$s_1=\int G^{(1)}(\cdot,y)s_2(y)d\alpha(y).$$

Then $s_1\in \mathbf{P}_1(X)$ by Theorem 18.3 in [4] and $(s_1, s_2)\in \mathbf{S}(X)$. This completes the proof.

3. Bi-harmonic Green's function

Let $(X, \mathbf{H})$ be an elliptic bi-harmonic space and $(X, \mathbf{H}_j)$ $(j=1, 2)$ be the associated Brelot's harmonic space. We suppose that $(X, \mathbf{H}_j)$ $(j=1, 2)$ satisfies the local proportionality axiom and $(X, \mathbf{H}_2)$ is strong. Then by Theorem 18.1 in [4] there exists the $\mathbf{H}_2$-Green's function $G^{(2)}(x, y)$ on X.

A function $N(x, y): X\times X\to(0, +\infty]$ is called the bi-harmonic Green's function of $(X, \mathbf{H})$ associated with $G^{(2)}(x, y)$ if for each $y\in X$,

$$(N(\cdot, y), G^{(2)}(\cdot\,, y))\in \mathbf{P}(X)\cap \mathbf{H}(X-\{y\}).$$

We shall show the following

Theorem 1. *There exists the bi-harmonic Green's function* $N(x, y)$ *of* $(X, \mathbf{H})$ *associated with* $G^{(2)}(x, y)$ *if and only if there exists the* $\mathbf{H}_1$*-Green's function* $G^{(1)}(x, y)$ *on* X *and, for each* $y\in X$,

$$\int G^{(1)}(x, z)G^{(2)}(z, y)d\alpha(z)<+\infty$$

for some $x\in X$, *where* α *is the composing measure of* $(X, \mathbf{H})$ *associated with* $G^{(1)}(x, y)$. *In this case we have*

$$N(x, y) = \int G^{(1)}(x, z)G^{(2)}(z, y)d\alpha(z).$$

Proof. The "only if" part is evident by Lemma 5, because for each $y\in X$, there exists $N(\cdot\,, y)$ such that $(N(\cdot\,, y), G^{(2)}(\cdot\,, y))\in \mathbf{P}(X)$.

To show the converse, put

$$p_y=\int G^{(1)}(\cdot, z)G^{(2)}(z, y)d\alpha(z)$$

for each $y\in X$. Then $p_y\in \mathbf{P}_1(X)$ by Theorem 18.3 in [4] and for any $x\in X$ and any $\mathbf{H}$-regular set $\omega\ni x$,

$$p_y(x) - \int p_y d\mu^\omega_x=\int G_\omega^{(1)}(x, z)G^{(2)}(z, y)d\alpha(y)$$

$$\geqq \int G^{(2)}(\cdot, y)d\nu^{\omega}_{x}.$$

Hence $(p_y, G^{(2)}(\cdot, y)) \in \mathbf{P}(X)$ by Corollary 5.16 in [6]. For any $x \in X-\{y\}$ and any $\mathbf{H}$-regular set ω with $x \in \omega \subset \bar{\omega} \subset X-\{y\}$ we have

$$p_y(x) - \int p_y d\mu^{\omega}_x = \int G_{\omega}^{(1)}(x, z) G^{(2)}(z, y) d\alpha(z)$$

$$= \int G_{\omega}^{(1)}(x, z) \left(\int G^{(2)}(\cdot, y) d\lambda^{\omega}_z\right) d\alpha(z)$$

$$= \int G^{(2)}(\cdot, y) d\nu^{\omega}_x.$$

Hence $(p_y, G^{(2)}(\cdot, y)) \in \mathbf{H}(X-\{y\})$ and so there exists the bi-harmonic Green's function of $(X, \mathbf{H})$ associated with $G^{(2)}(x, y)$.

Let $N(x, y)$ be the bi-harmonic Green's function of $(X, \mathbf{H})$ associated with $G^{(2)}(x, y)$. Since $N(\cdot, y) \in \mathbf{P}_1(X)$ for each $y \in X$, there exists a unique positive Radon measure m_y on X such that

$$N(\cdot, y) = \int G^{(1)}(\cdot, z) dm_y(z).$$

Since $(N(\cdot, y), G^{(2)}(\cdot, y)) \in \mathbf{H}(X-\{y\})$, we have $m_y = G^{(2)}(\cdot, y)\ \alpha$ on ω for any $\mathbf{H}$-regular set ω with $\omega \subset \bar{\omega} \subset X\text{-}\{y\}$. Hence we have

$$N(x, y) = \int G^{(1)}(x, z) G^{(2)}(z, y) d\alpha(z).$$

This completes the proof.

Remark. By Lemma 4 we know that $N(\cdot, y)$ is the pure hyperharmonic function of order 2 associated with $G^{(2)}(\cdot, y)$ for each $y \in X$.

Supposing the existence of the bi-harmonic Green's function, we shall give the integral represetation of an $\mathbf{H}$-potential on X.

Theorem 2. *Let (p_1, p_2) be an $\mathbf{H}$-potential on X. Then there exists a unique couple (m_1, m_2) of positive Rabon measures on X such that*

$$p_1(x) = \int N(x, y)\, dm_2(y) + \int G^{(1)}(x, y) dm_1(y),$$

$$p_2(x) = \int G^{(2)}(x, y) dm_2(y).$$

Proof. Let (p_1, p_2) be an $\mathbf{H}$-potential on X and q_1 be the pure hyperharmonic function of order 2 associated with p_2. Then there exists uniquely positive Radon measures β and m_2 such that

$$p_1(x) = \int G^{(1)}(x, y) d\beta(y),$$

$$p_2(x) = \int G^{(2)}(x, y) dm_2(y)$$

and by Lemma 4

$$q_1(x)=\int G^{(1)}(x, y)p_2(y)d\alpha(y).$$

Since (X, $\mathbf{H}_2$) is strong, there exists an increasing sequence $\{p_2^{(n)}\}_n$ of continuous $\mathbf{H}_2$-potentials on X such that $\lim_{n\to\infty} p_2^{(n)}=p_2$. By Corollary 5.16 in [6], $(p_1, p_2^{(n)})$ being in $\mathbf{P}(X)$, there exists the pure hyperharmonic function $q_1^{(n)}$ associated with $p_2^{(n)}$ and

$$q_1^{(n)}=\int G^{(1)}(\cdot, y)p_2^{(n)}(y)d\alpha(y).$$

Since $\Gamma(p_1-q_1^{(n)})\geqq p_2-p_2^{(n)}\geqq 0$ and $p_1-q_1^{(n)}\geqq q_1-q_1^{(n)}\geqq 0$, we have $p_1-q_1^{(n)}\in \mathbf{P}_1(X)$. Hence $\beta-p_2^{(n)}\alpha\geqq 0$ and so $\beta-p_2\alpha\geqq 0$. Letting $m_1=\beta-p_2\alpha$, we have

$$\begin{aligned} p_1(x)&=\int G^{(1)}(x, y)d\beta(y)\\ &=\int G^{(1)}(x, y)p_2(y)d\alpha(y)+\int G^{(1)}(x, y)dm_1(y)\\ &=\int G^{(1)}(x, z)\left(\int G^{(2)}(z, y)dm_2(y)\right)d\alpha(z)+\int G^{(1)}(x, y)dm_1(y)\\ &=\int N(x, y)dm_2(y)+\int G^{(1)}(x, y)dm_1(y). \end{aligned}$$

The uniqueness of this couple (m_1, m_2) of measures is evident by the uniqueness of the integral representation of an $\mathbf{H}_j$-potential p_j $(j=1, 2)$. This completes the proof.

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