A Certain Theorem on the Unitary Group $U(1, n; F)$

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Let $\boldsymbol{F}$ denote the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers, or the division ring of real quarternions $\boldsymbol{K}$. Let $V = V^{1,n}(\boldsymbol{F})$ denote the (right) vector space $\boldsymbol{F}^{n+1}$, together with the unitary structure defined by the $\boldsymbol{F}$-Hermitian form

$$\Phi(z, w) = -\bar{z}_0 w_0 + \bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_n w_n$$

for $z = (z_0, z_1, \cdots, z_n)^T$, $w = (w_0, w_1, \cdots, w_n)^T$ (where T denotes the transpose).

An automorphism g of V will be called unitary transformation. (g must be $\boldsymbol{F}$-linear and $\Phi(g(z), g(w)) = \Phi(z, w)$, for all $z, w \in V$.) We denote the group of all unitary transformations by $U(1, n; \boldsymbol{F})$.

Our purpose of this paper is to prove the following theorem.

THEOREM. *Let G be a discrete subgroup of $U(1, n; \boldsymbol{F})$. Let $a = (\beta_1, \beta_2, \cdots, \beta_n)^T \in H^n(\boldsymbol{F})$, either* (*i*) $\sum_{f_k \in G} (1 - \|f_k(a)\|) < \infty$, *or*

(*ii*) $\sum_{f_k \in G} (1 - \|f_k(a)\|) = \infty$

is independent of $a \in H^n(\boldsymbol{F})$, where $\|a\| = \{\sum_{j=1}^{n} |\beta_j|^2\}^{1/2}$.

G is called of convergence, or of divergence type, according as the case (i), or (ii).

1. Let us begin with recallig some notation and definitions. Let $V_- = \{z \in V : \Phi(z, z) < 0\}$. Obviously V_- is invariant under $U(1, n; \boldsymbol{F})$. Let $P(V)$ be the projective space obtained from V. This is defined as usual, by the equivalence relation in $V - \{0\}$: $u \sim v$ if there exists $\lambda \in \boldsymbol{F}^*$ (the multiplicative group in $\boldsymbol{F}$) such that $u = v\lambda$. $P(V)$ is the set of equivalence classes, with the quotient topology. Let P: $V - \{0\} \longrightarrow P(V)$ denote the projection map. We define: $H^n(\boldsymbol{F}) = P(V_-)$. If $g \in U(1, n; \boldsymbol{F})$, then $g(V_-) = V_-$ and $g(v\lambda) = g(v)\lambda$. Therefore $U(1, n; \boldsymbol{F})$ acts in $P(V)$, leaving $H^n(\boldsymbol{F})$ invariant.

2. Now we are ready to prove our theorem.

PROOF OF THEOREM. We denote $(1, 0, \cdots, 0)^T$ and $(\alpha_1, \alpha_2, \cdots, \alpha_{n+1})^T$ by O and

A, respectively. Let $P(O)=0$ and $P(A)=a$. Set

$$[A, B]=[|\Phi(A, B)|\{\Phi(A, A)\cdot\Phi(B, B)\}^{-\frac{1}{2}}]^{-1}$$

for $A, B\in V_{-}$. It is clear that $[A, B]$ is invariant under $U(1, n;\boldsymbol{F})$. Let

$$f_k=\begin{pmatrix} a_{1,1}^{(k)} & a_{1,2}^{(k)} & \cdots & a_{1,n+1}^{(k)} \\ & \cdots & & \\ a_{n+1,1}^{(k)} & a_{n+1,2}^{(k)} & \cdots & a_{n+1,n+1}^{(k)} \end{pmatrix}.$$

We note that

$$f_k(A)=(\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j, \sum_{j=1}^{n+1}a_{2,j}^{(k)}\alpha_j, \cdots, \sum_{j=1}^{n+1}a_{n+1,j}^{(k)}\alpha_j),$$

$$f_k(O)=(a_{1,1}^{(k)}, a_{2,1}^{(k)}, \cdots, a_{n+1,1}^{(k)}),$$

$$\Phi(f_k(A), f_k(O))=-\overline{\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j}\, a_{1,1}^{(k)}+\sum_{m=2}^{n+1}(\overline{\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j})a_{m,1}^{(k)},$$

$$\Phi(f_k(A), f_k(A))=-|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^2+\sum_{m=2}^{n+1}|\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j|^2,$$

$$\Phi(f_k(O), f_k(O))=-|a_{1,1}^{(k)}|^2+\sum_{m=2}^{n+1}|a_{m,1}^{(k)}|^2,$$

$$||P(f_k(A))||^2=\sum_{m=2}^{n+1}(|\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j|^2\cdot|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^{-2}),$$

$$||P(f_k(O))||^2=\sum_{m=2}^{n+1}|a_{m,1}^{(k)}|^2\cdot|a_{1,1}^{(k)}|^{-2},$$

$$1-||P(f_k(A))||^2=\{(|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^2-\sum_{m=2}^{n+1}|\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j|^2)\,|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^{-2}$$

$$1-|P(f_k(O))|^2=(|a_{1,1}^{(k)}|^2-\sum_{m=2}^{n+1}|a_{m,1}^{(k)}|^2)\,|a_{1,1}^{(k)}|^{-2}, \text{ and}$$

$$||P(f_k(A)||\cdot||P(f_k(O))||=[\sum_{m=2}^{n+1}(|\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j|^2\cdot|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^{-2}) \sum_{m=2}^{n+1}(|a_{m,1}^{(k)}|^2\cdot|a_{1,1}^{(k)}|^{-2})]^{\frac{1}{2}}.$$

Using the above, we have

$$\begin{aligned} 1-||a||^2 &= [A, O]^2=[f_k(A), f_k(O)]^2 \\ &= \{\Phi(f_k(A), f_k(A))\cdot\Phi(f_k(O), f_k(O))\}\,|\Phi(f_k(A), f_k(O))|^{-2} \\ &= \{(-|\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j|^2+\sum_{m=2}^{n+1}|\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j|^2) \\ &\quad (-|a_{1,1}^{(k)}|^2+\sum_{m=2}^{n+1}|a_{m,1}^{(k)}|^2) \\ &\quad |-\overline{\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j}\,a_{1,1}^{(k)}+\sum_{m=2}^{n+1}(\overline{\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j})a_{m,1}^{(k)}|^{-2} \\ &= (1-||P(f_k(A))||^2)\,(1-||P(f_k(O))||^2) \\ &\quad |1-\sum_{m=2}^{n+1}(\overline{\sum_{j=1}^{n+1}a_{m,j}^{(k)}\alpha_j}\,a_{m,1}^{(k)}/\overline{\sum_{j=1}^{n+1}a_{1,j}^{(k)}\alpha_j}\,a_{1,1}^{(k)})|^{-2} \end{aligned}$$

$$\leqq (1+\|P(f_k(A))\|)(1-\|P(f_k(A))\|)(1+\|P(f_k(O))\|)(1-\|P(f_k(O))\|)$$

$$(1-\sum_{m=1}^{n+1}|\sum_{j=1}^{n+1} a_{m,j}^{(k)}\alpha_j a_{m,1}^{(k)} / \sum_{j=1}^{n+1} a_{1,j}^{(k)}\alpha_j a_{1,1}^{(k)}|)^{-2}$$

$$\leqq 4(1-\|P(f_k(A))\|)(1-\|P(f_k(O))\|)(1-\|P(f_k(A))\|\cdot\|P(f_k(O))\|)^{-2}$$

$$\leqq \begin{cases} 4(1-\|P(f_k(A))\|)(1-\|P(f_k(O))\|)^{-1}. \\ 4(1-\|P(f_k(O))\|)(1-\|P(f_k(A))\|)^{-1}. \end{cases}$$

Therefore we see that

$$\frac{1}{4}(1-\|a\|^2)(1-\|P(f_k(O))\|) \leqq 1-\|P(f_k(A))\|$$

$$\leqq 4(1-\|P(f_k(O))\|)(1-\|a\|^2)^{-1}.$$

Thus our theorem is completely proved.

References

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