On Extensions of the Contraction Principle

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In this paper we shall extend the results of Okano [3] and Edelstein [1].

1. Throughout this paper, a "ranked space" means a ranked space defined in [3]. Also we shall conform ourselves to the notions, the terminology and the notations of [3]. But in this paper, for convenience' sake, we shall denote by $V(p;\varepsilon)$, $U(p;\varepsilon)$ etc. the neighborhoods of a point p of rank ε and assume that the sequences are indexed by the set $\{0, 1, 2, \dots\}$.

2. Strictly speaking, a ranked space S is expressed as the triple $(S, \mathfrak{N}(p), \mathfrak{N}_\varepsilon)$, where S is the underlying set, $\mathfrak{N}(p)$ is the neighborhoods system of $p(p\in S)$ and $\mathfrak{N}_\varepsilon$ is the family of neighborhoods of rank $\varepsilon(0<\varepsilon<\infty)$.

Let L be an indexing set and $\{(R, \mathfrak{V}^\lambda(p), \mathfrak{V}_\varepsilon^\lambda)\}$ $(\lambda\in L)$ a family of ranked spaces (the underlying sets are identically R for all ranked spaces of the family); we denote by $V^\lambda(p;\varepsilon)$, $U^\lambda(p;\varepsilon)$ etc. the neighborhoods of $p(p\in R)$ which belong to $\mathfrak{V}_\varepsilon^\lambda$ $(\lambda\in L)$. Now, for each $p\in R$ and each positive number ε, put $\mathfrak{V}(p)=\bigcup_{\lambda\in L}\mathfrak{V}^\lambda(p)$ and $\mathfrak{V}_\varepsilon=\bigcup_{\lambda\in L}\mathfrak{V}_\varepsilon^\lambda$. Then $(R, \mathfrak{V}(p), \mathfrak{V}_\varepsilon)$ becomes a ranked space: this space is called the *ranked union space* of $\{(R, \mathfrak{V}^\lambda(p), \mathfrak{V}_\varepsilon^\lambda)\}$ $(\lambda\in L)$ (cf. Nakanishi [2]).

Throughout this section, for simplicity, we agree that the capital R denotes the ranked space $(R, \mathfrak{V}(p), \mathfrak{V}_\varepsilon)$ defined above, i.e., R is the ranked union space of $\{(R, \mathfrak{V}_\lambda(p), \mathfrak{V}_\varepsilon^\lambda)\}$ $(\lambda\in L)$.

Let $\{\sigma_\lambda\}$ $(\lambda\in L)$ be a family of positive numbers or ∞. For each $a\in R$ and each $\lambda\in L$, we denote by $D^\lambda(a;\sigma_\lambda)$ the union of all neighborhoods $V^\lambda(a;\varepsilon)$ with $\varepsilon<\sigma_\lambda$; and we put $D(a;\{\sigma_\lambda\})=\bigcup_{\lambda\in L}D^\lambda(a;\sigma_\lambda)$.

DEFINITION 1. Let $\{\eta_\lambda\}$, $\{k_\lambda\}$ $(\lambda \in L)$ be families of constants or ∞ such that $0<\eta_\lambda \leq \infty$, $0<k_\lambda<1$ respectively. A mapping f of R into itself is called an $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction if, for each $\lambda \in L$, the following is valid: For any point y belonging to some $V^\lambda(x;\varepsilon)$ with $\varepsilon<\eta_\lambda$, there corresponds a $U^\lambda(f(x); k_\lambda \varepsilon)$ which contains $f(y)$. And an $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction is said to be α-type or β-type according as $\inf_{\lambda \in L} \eta_\lambda > 0$ or $\inf_{\lambda \in L} \eta_\lambda = 0$ respectively.

DEFINITION 2 (due to [3]). We say that R satisfies (m) if the following is valid:

(m) For a sequence of points $\{x_n\}$ such that x_{n+1} belongs to some neighborhood of x_n of rank ε_n, if $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, then there exists a Cauchy sequence $\{V_n^{\lambda(n)}(x_n;\varepsilon_n)\}$.

THEOREM 1. *Let R be s-complete and s-separated, and satisfy* (m). *And let f be an (ss^*)-continuous α-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. If there exists a point a such that $f(a) \in D(a; \{\eta_\lambda\})$, then f has at least one fixed point x in $\bigcup_{n=0}^{\infty} D(f^n(a); \{\eta_\lambda\})$ and we have $x = s\text{-}\lim_{n\to\infty} f^n(a)$, where $f^0(a)=a$ and $f^n(a)=f(f^{n-1}(a))$.*

PROOF. The proof of an existence of a point x such that $f(x)=x$ and that $x = s\text{-}\lim_{n\to\infty} f^n(a)$ is the same as that of Theorem 2 in [3], and so is omitted. Now since $x = s\text{-}\mathrm{ilm}_{n\to\infty} f^n(a)$ and since $\inf_{\lambda\in L} \eta_\lambda > 0$, it is clear that there is some $V^\nu(f^{n'}(a);\varepsilon)$ such that $x \in V^\nu(f^{n'}(a);\varepsilon)$ and $\varepsilon<\eta_\nu$. Thus $x \in \bigcup_{n=0}^{\infty} D(f^n(a); \{\eta_\lambda\})$.

REMARK 1. The statement of Theorem 1 is valid for a β-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction except that $x \in \bigcup_{n=0}^{\infty} D(f^n(a); \{\eta_\lambda\})$.

DEFINITION 3 (due to [3]). We say that R satisfies (t) if the following is valid:

(t) To every pair of distinct points p, q there corresponds a positive number $\rho(p,q)$ such that, if $p \in U^\lambda(r;\varepsilon)$ and $q \in V^\nu(r;\delta)$ for some r, then $\varepsilon+\delta>\rho(p,q)$.

PROPOSITION 1. *Let R be complete and satisfy* (m) *and* (t). *Then each α-type $\{(\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself is contiouous with respect to the c^*-convergence.*

PROOF (cf. Proof of Prop. 2 in [3]). Let f be an α-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. It is sufficient to show that $c\text{-}\lim_{n\to\infty} x_n = x$ implies $c\text{-}\lim_{i\to\infty} f(x_{n(i)}) = f(x)$ for some subsequence $\{n(i)\}$ of $\{n\}$. Now since $\inf_{\lambda\in L} \eta_\lambda > 0$, from $c\text{-}\lim_{n\to\infty} x_n = x$, there exist a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$ and a Cauchy sequence $\{V_i^{\lambda(i)}(x_{n(i)};\varepsilon_i)\}$ such that $x\in\bigcap_{i=0}^{\infty}V_i^{\lambda(i)}(x_{n(i)};\varepsilon_i)$, $\sum_{i=0}^{\infty}\varepsilon_i<\infty$ and that $\varepsilon_i<\eta_{\lambda(i)}$ for all i. Hence, for each i, there exist $U_i^{\lambda(i)}(f(x_{n(i)});k_{\lambda(i)}\varepsilon_i)$ and $U^{*}{}_i^{\lambda(i)}(f(x_{n(i)});k_{\lambda(i)}\varepsilon_i)$ which contain $f(x)$ and $f(x_{n(i+1)})$ respectively, for x and $x_{n(i+1)}$ belong to $V_i^{\lambda(i)}(x_{n(i)};\varepsilon_i)$. As $\sum_{i=0}^{\infty}k_{\lambda(i)}\varepsilon_i\leq\sum_{i=0}^{\infty}\varepsilon_i<\infty$, there is by (m) a Cauchy sequence $\{W_i^{\nu(i)}(f(x_{n(i)});\delta_i)\}$. So, from the completeness of R, there is a point y such that $y\in\bigcap_{i=0}^{\infty}W_i^{\nu(i)}(f(x_{n(i)});\delta_i)$; that is $c\text{-}\lim_{i\to\infty}f(x_{n(i)})=y$. But since $\lim_{i\to\infty}(k_{\lambda(i)}\varepsilon_i+\delta_i)=0$, we have $y=f(x)$ by (t).

THEOREM 2. *Let R be complete and satisfy* (m) *and* (t). *And let f be an α-type $(\{\eta_\lambda\},\{k_\lambda\})$-cnotraction of R into itself. If there exists a point a such that $f(a)\in D(a;\{\eta_\lambda\})$, then f has a unique fixed point x in $\bigcup_{n=0}^{\infty}D(f^n(a);\{\eta_\lambda\})$ and we have $x=c\text{-}\lim_{n\to\infty}f^n(a)$.*

PROOF. Since R satisfies (t), it is c-separated; and so the statement of this theorem follows from Theorem 1, Proposition 1, Proposition 3 in [3] and Lemma in [3] except the uniqueness of the fixed point. Now let y be a fixed point of f in $\bigcup_{n=0}^{\infty}D(f^n(a);\{\eta_\lambda\})$. Then there is some $V^{\nu}(f^{n'}(a);\varepsilon)$ such that $y\in V^{\nu}(f^{n'}(a);\varepsilon)$ and $\varepsilon<\eta_\nu$. Hence, for each n, there exists a $U_n^{\nu}(f^{n'+n}(a);k_\nu^n\varepsilon)$ which contains y. As $\lim_{n\to\infty}k_\nu^n\varepsilon=0$ and $x=c\text{-}\lim_{n\to\infty}f^n(a)$, we have $y=x$ by (t).

REMARK 2. The statements of Proposition 1 and Theorem 2 are not necessarily valid for β-type $(\{\eta_\lambda\},\{k_\lambda\})$-contractions respectively.

DEFINITION 4. We say that R satisfies $(\tilde{\text{m}})$ if the following is valid:

$(\tilde{\text{m}})$ For a sequence of points $\{x_n\}$ such that x_{n+1} belongs to some $V_n^{\lambda(n)}(x_n;\varepsilon_n)$, if $\sum_{n=0}^{\infty}\varepsilon_n<\infty$ and if the set $\{\lambda(n);\ n=0,1,2,\ldots\}$ is finite, then there exists a Cauchy sequence $\{U_n^{\nu(n)}(x_n;\delta_n)\}$ such that the set $\{\nu(n);\ n=0,1,2,\ldots\}$ is finite.

THEOREM 3. *Let R be s-cmplete and s-separated, and satisfy $(\tilde{\text{m}})$. And let f be an (ss^*)-continious $(\{\eta_\lambda\},\{k_\lambda\})$-contraction of R into itself. If there exists a point a such that $f(a)\in D(a;\{\eta_\lambda\})$, then f has at least one fixed point x in $\bigcup_{n=0}^{\infty}D(f^n(a);\{\eta_\lambda\})$ and we have $x=s\text{-}\lim_{n\to\infty}f^n(a)$.*

PROOF. Taking account of the proof of Theorem 2 in [3], it is sufficient to show an existence of a Cauchy sequence $\{W_n^{\lambda(n)}(f^n(a);\delta_n)\}$ such that $\delta_{n'}<\eta_{\lambda(n')}$

for some n'. Now there is some $V^{\nu}(a;\varepsilon)$ such that $f(a)\in V^{\nu}(a;\varepsilon)$ and $\varepsilon<\eta_{\nu}$. Hence, for each n, there exists a $U_n^{\nu}(f^n(a);k_{\nu}^n\varepsilon)$ which contains $f^{n+1}(a)$. As $\sum_{n=0}^{\infty} k_{\nu}^n\varepsilon<\infty$, there is by ($\widetilde{\mathrm{m}}$) a Cauchy sequence $\{W_n^{\lambda(n)}(f^n(a);\delta_n)\}$ such that the set $\{\lambda(n);n=0,1,2,\dots\}$ is finite. Clearly $\delta_{n'}<\eta_{\lambda(n')}$ for some n'.

THEOREM 4. *Let R be complete and satisfy ($\widetilde{\mathrm{m}}$) and* (t). *And let f be an $(\{\eta_\lambda\},\{k_\lambda\})$-contraction of R into itself. If there exists a point a such that $f(a)\in D(a;\{\eta_\lambda\})$, then f has a unique fixed point x in $\bigcup_{n=0}^{\infty} D(f^n(a);\{\eta_\lambda\})$ and we have $x=c\text{-}\lim_{n\to\infty} f^n(a)$.*

PROOF. Taking account of the proofs of Theorem 2 in [3] and Theorem 2, it is sufficient to show an existence of a point x such that $x\in\bigcup_{n=0}^{\infty} D(f^n(a);\{\eta_\lambda\})$, $c\text{-}\lim_{n\to\infty} f^n(a)=x$ and that $c\text{-}\lim_{i\to\infty} f^{n(i)}(a)=f(x)$ for some subsequence $\{n(i)\}$ of $\{n\}$. Now, by the same argument as in the proof of Theorem 3, we shall have a Cauchy sequence $\{W_n^{\lambda(n)}(f^n(a);\delta_n)\}$ such that the set $\{\lambda(n);n=0,1,2,\dots\}$ is finite. So, from the completeness of R, there is a point x such that $x\in\bigcup_{n=0}^{\infty} W_n^{\lambda(n)}(f^n(a);\delta_n)$. Clearly $x\in\bigcup_{n=0}^{\infty} D(f^n(a);\{\eta_\lambda\})$ and $c\text{-}\lim_{n\to\infty} f^n(a)=x$; and since the set $\{\lambda(n);n=0,1,2,\dots\}$ is finite, by the argument similar to the proof of Proposition 1, we shall have $c\text{-}\lim_{i\to\infty} f^{n(i)}(a)=f(x)$ for some subsequence $\{n(i)\}$ of $\{n\}$.

REMARK 3. In Theorems 3 and 4, R need not be complete: It is sufficient that $(R,\mathfrak{B}^{\lambda}(p),\mathfrak{B}_{\varepsilon}^{\lambda})$ is cmplete for each $\lambda\in L$.

DEFINITION 5. Let $\{\tau_\lambda\}$ $(\lambda\in L)$ be a family of positive numbers or ∞. By a $\{\tau_\lambda\}$-chain in R we mean a finite family of neighborhoods $\{V_i^{\lambda(i)}(p_i;\varepsilon_i)\}$ $(i=0,\dots,n)$ such that $\varepsilon_i<\tau_{\lambda(i)}$ for all i and that $p_{i+1}\in V_i^{\lambda(i)}(p_i;\varepsilon_i)$ for each $i=0,\dots,n-1$. And, for points p and q, we say that p is $\{\tau_\lambda\}$-chainable to q if there exists a $\{\tau_\lambda\}$-chain $\{U_i^{\nu(i)}(r_i;\delta_i)\}$ $(i=0,\dots,l)$ such that $r_0=p$ and $q\in U_l^{\nu(l)}(r_l;\delta_l)$. Moreover we say that R is $\{\tau_\lambda\}$-chainable (resp. $\{\tau_\lambda\}$-bichainable) if, for any pair of points p, q, at least one of which is $\{\tau_\lambda\}$-chainable to the other (resp. p is $\{\tau_\lambda\}$-chainable to q and vice versa).

Considering $\mathrm{card}(L)=1$ ($\mathrm{card}(L)$ denotes the power of L), for every τ, $0<\tau\leq\infty$, the ranked space of Example 2 in [3] is $\{\tau\}$-chainable and the ranked spaces of Examples 1, 3 and 4 in [3] are $\{\tau\}$-bichainable.

For each $a\in R$ and each family $\{\tau_\lambda\}$ $(\lambda\in L)$ of positive numbers or ∞, we put $D^*(a;\{\tau_\lambda\})=\{p\in R;a$ is $\{\tau_\lambda\}$-chainable to $p\}$.

In order to extend Theorem in [1], we introduce the following

DEFINITION 6. We say that R satisfies (M*) if the following is valid:

(M*) There exists a positive constant C such that: If, for points p and q, there exsist a finite family of neighborhoods $\{V_i^{\lambda(i)}(r_i;\varepsilon_i)\}$ $(i=0,\dots,n)$ such that $r_0=p$, $q\in V_n^{\lambda(n)}(r_n;\varepsilon_n)$ and that $r_{i+1}\in V_i^{\lambda(i)}(r_i;\varepsilon_i)$ for each $i=0,\dots,n-1$, then, for any positive number ε, there is some $U^\nu(p;\delta)$ such that $q\in U^\nu(p;\delta)$ and $\delta\leq C\sum_{i=0}^{n}\varepsilon_i+\varepsilon$.

Considering card$(L)=1$, the ranked spaces of Examples 1—4 in [3] satisfy (M*). Also premetric spaces of [3] and partially ordered sets, considered as ranked spaces (of card$(L)=1$) (see [3]) respectively, satisfy (M*).

LEMMA 1. *Let R satisfy* (M*) *and let f be an* $(\{\eta_\lambda\},\{k_\lambda\})$*-contraction of R into itself. If $q\in D^*(p;\{\eta_\lambda\})$, then there exists a sequence of neighborhoods* $\{W_n^{\nu(n)}(f^n(p);\delta_n)\}$ *such that* $f^n(q)\in W_n^{\nu(n)}(f^n(p);\delta_n)$ *and* $\lim_{n\to\infty}\delta_n=0$.

PROOF. There exists an $\{\eta_\lambda\}$-chain $\{V_i^{\lambda(i)}(r_i;\varepsilon_i)\}$ $(i=0,\dots,l)$ such that $r_0=p$ and $q\in V_l^{\lambda(l)}(r_l;\varepsilon_l)$. Hence, for each pair (n,i) $(n=0,1,2,\dots;\ i=0,\dots,l-1)$, there is some $U_{n,i}^{\lambda(i)}(f^n(r_i);k^n_{\lambda(i)}\varepsilon_i)$ which contains $f^n(r_{i+1})$. So, for each n, there exists by (M*) a $W_n^{\nu(n)}(f^n(p);\delta_n)$ such that $f^n(q)\in W_n^{\nu(n)}(f^n(p);\delta_n)$ and $\delta_n\leq C\sum_{i=0}^{l}k^n_{\lambda(i)}\varepsilon_i+2^{-n}$. Clearly $\lim_{n\to\infty}\delta_n=0$.

THEOREM 5. *Let R be s-complete and s-separated, and satisfy* (m) *and* (M*). *And let f be an* (ss^*)*-continuous α-type* $(\{\eta_\lambda\},\{k_\lambda\})$*-contraction of R into itself. If there exists a point a such that $f(a)\in D^*(a;\{\eta_\lambda\})$, then f has at least one fixed point x in $D^*(a;\{\eta_\lambda\})$ and we have* $x=s\text{-}\lim_{n\to\infty}f^n(a)$.

PROOF. Since $\inf_{\lambda\in L}\eta_\lambda>0$, by Lemma 1, there exist a non-negative integer N and a neighborhood $W^\nu(f^N(a);\delta)$ such that $f^{N+1}(a)\in W^\nu(f^N(a);\delta)$ and $\delta<\eta_\lambda$. Thus $f(a_0)\in D(a_0;\{\eta_\lambda\})$, where $a_0=f^N(a)$. Hence, by Theorem 1, f has at least one fixed point x in $\bigcup_{n=0}^{\infty}D(f^n(a_0);\{\eta_\lambda\})$ and $x=s\text{-}\lim_{n\to\infty}f^n(a_0)$. But since $f(a)\in D^*(a;\{\eta_\lambda\})$ and since f is an $(\{\eta_\lambda\},\{k_\lambda\})$-contraction, we have $f^{n+1}(a)\in D^*(f^n(a);\{\eta_\lambda\})$ for each n, and so $f^n(a)\in D^*(a;\{\eta_\lambda\})$ for all n. Therefore $x=s\text{-}\lim_{n\to\infty}f^n(a)$ from $x=s\text{-}\lim_{n\to\infty}f^n(a_0)$ and (m); and we have $x\in\bigcup_{n=0}^{\infty}D(f^n(a_0);\{\eta_\lambda\})\subset D^*(a;\{\eta_\lambda\})$.

REMARK 4. The statement of Theorem 5 is not necessarily valid for a β-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction even if we except that $x\in D^*(a;\{\eta_\lambda\})$.

LEMMA 2. *Let R sasisfy* (M*) *and* (t), *and let f be an α-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. And let x, y be fixed points of f. If both x and y are contained in some $D^*(p;\{\eta_\lambda\})$, then $x=y$.*

PROOF. Since $\inf_{\lambda\in L}\eta_\lambda>0$, by Lemma 1, there exist non-negative integers N, N' and neighborhoods $W^\lambda(f^N(p);\varepsilon)$, $W^\nu(f^{N'}(p);\delta)$ such that $x\in W^\lambda(^N(p);\varepsilon)$, $y\in W^\nu(f^{N'}(p);\delta)$, $\varepsilon<\eta_\lambda$ and $\delta<\eta_\nu$. Hence we easily have $x=y$ by (t), since f is an $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction (cf. Proof of Th. 2).

THEOREM 6. *Let R be complete and satisfy* (m), (M*) *and* (t). *And let f be an α-type $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. If there exists a point a such that $f(a)\in D^*(a;\{\eta_\lambda\})$, then f has a unique fixed point x in $D^*(a;\{\eta_\lambda\})$ and we have $x=c\text{-}\lim_{n\to\infty} f^n(a)$.*

PROOF. Since R satisfies (t), it is c-separated; and so this theorem follows from Theorem 5, Proposition 1, Proposition 3 in [3], Lemma in [3] and Lemma 2.

DEFINITION 7. We say that R satisfies $(\widetilde{M}^*)$ if the following is valid:

$(\widetilde{M}^*)$ There exists a positive constant C such that: If, for points p and q, there exists a finite family of neighborhoods $\{V_i^{\lambda(i)}(r_i;\varepsilon_i)\}$ $(i=0,\dots,n)$ such that $r_0=p$, $q\in V_n^{\lambda(n)}(r_n;\varepsilon_n)$ and that $r_{i+1}\in V_i^{\lambda(i)}(r_i;\varepsilon_i)$ for each $i=0,\dots,n-1$, then for each positive number ε, there is some $U^\nu(p;\delta)$ such that $\nu\in\{\lambda(i); i=0,\dots,n\}$, $q\in U^\nu(p;\delta)$ and that $\delta\leq C\sum_{i=0}^{n}\varepsilon_i+\varepsilon$.

By virtue of $(\widetilde{M}^*)$, the argument similar to the proof of Lemma 1 will provide the following

LEMMA 3. *Let R satisfy* $(\widetilde{M}^*)$ *and let f be an $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. If $q\in D^*(p;\{\eta_\lambda\})$, then there exists a sequence of neighborhoods* $\{W_n^{\nu(n)}(f^n(p);\delta_n)\}$ such that $f^n(q)\in W_n^{\nu(n)}(f^n(p);\delta_n)$, $\lim_{n\to\infty}\delta_n=0$ and that the set $\{\nu(n); n=0,1,2,\dots\}$ is finite.

Using Lemma 3, by the argument similar to the proof of Lemma 2, we shall have the following

LEMMA 4. *Let R satisfy* ($\widetilde{\mathrm{M}}$*) *and* (t), *and let f be an* $(\{\eta_\lambda\}, \{k_\lambda\})$*-contraction of R into itself. And let x, y be fixed points of f. If both x and y are contained in some* $D^*(p; \{\eta_\lambda\})$, *then* $x=y$.

And using Theorem 3 and Lemma 3, by the argument similar to the proof of Theorem 5, we shall have the following

THEOREM 7. *Let R be s-complete and s-separated, and satisfy* ($\widetilde{\mathrm{m}}$) *and* ($\widetilde{\mathrm{M}}$*). *And let f be an (ss*)-continuous* $(\{\eta_\lambda\}, \{k_\lambda\})$*-contraction of R into itself. If there exists a point a such that* $f(a)\in D^*(a; \{\eta_\lambda\})$, *then f has at least one fixed point x in* $D^*(a; \{\eta_\lambda\})$ *and we have* $x=s\text{-}\lim_{n\to\infty} f^n(a)$.

Moreover using Theorem 4 and Lemmas 3, 4, by the argument similar to the proof of Theorem 5, we shall have the following

THEOREM 8. *Let R be complete and satisfy* ($\widetilde{\mathrm{m}}$), ($\widetilde{\mathrm{M}}$*) *and* (t). *And let f be an* $(\{\eta_\lambda\}, \{k_\lambda\})$*-contraction of R into itself. If there exists a point a such that* $f(a)\in D^*(a; \{\eta_\lambda\})$, *then f has a unique fixed point x in* $D^*(a; \{\eta_\lambda\})$ *and we have* $x=c\text{-}\lim_{n\to\infty} f^n(a)$.

DEFINITION 8. Let $\{\eta_\lambda\}$, $\{k_\lambda\}$ $(\lambda\in L)$ be families of constants or ∞ such that $0<\eta_\lambda\leq\infty$, $0<k_\lambda<1$ respectively; let $\{N_\lambda\}$ $(\lambda\in L)$ be a family of positive integers. A mapping f of R into itself is called an $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction if, for each $\lambda\in L$, the following is valid: For any point y belonging to some $V^\lambda(x; \varepsilon)$ with $\varepsilon<\eta_\lambda$, there corresponds a $U^\lambda(f^{N_\lambda}(x); k_\lambda\varepsilon)$ which contains $f^{N_\lambda}(y)$. And an $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction is said to be α-type (resp. α^*-type) if $\inf_{\lambda\in L}\eta_\lambda>0$ (resp. the set $\{N_\lambda; \lambda\in L\}$ is finite); and "$(\alpha\alpha^*)$-type" means both α-type and α^*-type.

LEMMA 5. *Let R satisfy* (M*) *and* (t), *and let f be a mapping of R into itself such that, for some positive integer N, f^N is an α-type* $(\{\eta_\lambda\}, \{k_\lambda\})$*-contraction ($f^N$ denotes the N times iteration of f). If R is* $\{\eta_\lambda\}$*-chainable and if f^N has a fixed point x, then x is the unique fixed point of f in R.*

PROOF. Since R is *is* $\{\eta_\lambda\}$-chainable, by Lemma 2, x is the unique fixed point of f^N in R. Hence $f^N(f(x))=f(f^N(x))=f(x)$ must imply $f(x)=x$. But since any fixed point y of f is also a fixed point of f^N, we have $y=x$ by the uniqueness of the fixed point of f^N. Thus x is the unique fixed point of f in R.

Using Lemma 4, by the argument similar to the proof of Lemma 5, we shall have the following

LEMMA 6. *Let R satisfy $(\widetilde{\mathrm{M}}^*)$ and* (t), *and let f be a mapping of R into itself such that, for some positive integer N, f^N is an $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction. If R is $\{\eta_\lambda\}$-chainable and if f^N has a fixed point x, then x is the unique fixed point of f in R.*

PROPOSITION 2. *Let f be an α^*-type $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. Then there exists a positive integer N^* for which f^{N^*} is an $(\{\eta_\lambda\}, \{k_\lambda^*\})$-contraction.*

PROOF. Since the set $\{N_\lambda;\ \lambda\in L\}$ is finite, let its elements be $N_0^*, \dots, N_n^*$. Putting $N^*=N_0^*\times\cdots\times N_n^*$ and $k_\lambda^*=k_\lambda^{N^*N_\lambda^{-1}}$ $(\lambda\in L)$, it will be easily seen that f^{N^*} is the $(\{\eta_\lambda\}, \{k_\lambda^*\})$-contraction.

COROLLARY 1. *Let R be complete and satisfy* (m), (M*) *and* (t). *And let f be an $(\alpha\alpha^*)$-type $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. If R is $\{\eta_\lambda\}$-bichainable, then f has a unique fixed point x in R and, for some positive integer N, we have $x=c\text{-}\lim_{n\to\infty} f^{Nn}(a)$ for any $a\in R$.*

COROLLARY 2. *Let R be complete and satisty $(\widetilde{\mathrm{m}})$, $(\widetilde{\mathrm{M}}^*)$ and* (t). *And let f be an α^*-type $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself. If R is $\{\eta_\lambda\}$-bichainable, then f has a unique fixed point x in R and, for some positive integer N, we have $x=c\text{-}\lim_{n\to\infty} f^{Nn}(a)$ for any $a\in R$.*

REMARK 5. In Corollaries 1 and 2, $x=c\text{-}\lim_{n\to\infty} f^{Nn}(a)$ for any $a\in R$ does not necessarily imply $x=c\text{-}\lim_{n\to\infty} f^n(a)$ (for some $a\in R$) but implies $x=c^*\text{-}\lim_{n\to\infty} f^n(a)$ for any $a\in R$.

To construct the theorems analogous to Corollaries 1 and 2 for non-α^*-type $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contractions seems to be difficult. For example, the following statement is not necessarily valid.

"Let R be complete and satisfy ($\widetilde{\text{m}}$), ($\widetilde{\text{M}}$*) and (t). And let f be a non-α^*-type $\{N_\lambda\}$-iterative $(\{\eta_\lambda\}, \{k_\lambda\})$-contraction of R into itself which is continuous with respect to the c^*-convergence. If R is $\{\eta_\lambda\}$-bichainable and if there exist a point a and a neighborhood $V^\lambda(a; \varepsilon)$ such that $f^{N_\lambda}(a) \in V^\lambda(a; \varepsilon)$ and $\varepsilon < \eta_\lambda$, then f has a fixed point."

Note. Let $\{\eta_\lambda\}$, $\{k_\lambda\}$ $(\lambda \in L)$ be the same as in Definition 1 respectively. A *generalized* $(\{\eta_\lambda\}, \{k_\lambda\})$ *-contraction* f may be defined as follows: f is a mapping of R into itself such that, for any point y belonging to some $V^\lambda(x; \varepsilon)$ with $\varepsilon < \eta_\lambda$, there corresponds a $U^\nu(f(x); k_\nu\varepsilon)$ which contains $f(y)$.

To extend the results of this paper for generalized $(\{\eta_\lambda\}, \{k_\lambda\})$-contractions is interesting but seems to be more complicated.

References

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