On Star Convergences of r-Convergences in Ranked Spaces

Masato HIKIDA

Department of Applied Mathematics, Okayama University of Science
Ridai-cho, Okayama 700, Japan

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Prof. K. Kunugi has given the definitions of the ranked space and convergences in this space (see [1]). Fundamental one of convergences is what is called the r-convergence. In a ranked space of indicator ω_0 the r-convergence is Fréchet's L-convergence as long as every sequence is r-convergent to at most one point, but generally is not L^*-convergence. In this paper, for a ranked space with some property, we will give a necessary and sufficient condition for which the r-convergence is the star convergence.

Throughout this paper a "ranked space" means a ranked space of indicator ω_0. Terminologies and notations which will be used in this paper are stated in Appendix.

1. Let R be a ranked space and p be a point of R.

DEFINITION. Let $\{p_i\}$ $(i=0, 1, 2, \ldots)$ be a sequence of points of R and $V=\{v_i\}\in \boldsymbol{F}(p)$. If for any v_i there is an integer $i'\geq 0$ such that $p_j\in v_i$ for all $j\geq i'$, then we say that $\{p_i\}$ is *r-convergent to p with respect to V* and denote this by $p_i \xrightarrow[r]{} p(V)$. Also, $\{p_i\}$ is said to be *r-convergent to p*, denoted $p_i \xrightarrow[r]{} p$, if $p_i \xrightarrow[r]{} p(V)$ for some $V\in \boldsymbol{F}(p)$.

DEFINITION. If $p_i \underset{r}{\not\to} p$ (i. e. it is false that $p_i \xrightarrow[r]{} p$), then there is a subsequence $\{s_j\}$ of $\{p_i\}$ such that for any subsequence $\{t_k\}$ of $\{s_j\}$, $t_k \underset{r}{\not\to} p$, we say that $\{p_i\}$ is *r-star convergent to p* or an *r-convergence to p* of $\{p_i\}$ is a *star convergence.*

DEFINITION. A subset S of $\boldsymbol{F}(p)$ is said to be *essentially countable* if the quotient set $S/\sim$ is countable.

DEFINITION. A subset S of $\boldsymbol{F}(p)$ is said to be *cofinal in $\boldsymbol{F}(p)$* if for any $V\in \boldsymbol{F}(p)$ there is an $U\in S$ such that $V<U$.

THEOREM. *If there exists an essentially countable subset $\boldsymbol{S}$ of $\boldsymbol{F}(p)$ which is cofinal in $\boldsymbol{F}(p)$, then the r-convergence to p is the star convergence if and only if $\boldsymbol{S}$ is directed (i. e. for any two V, $U\in \boldsymbol{S}$ there is a $W\in \boldsymbol{S}$ such that $V<W$ and $U<W$).*

PROOF. (a) The "if" part. Suppose that there exists a sequence $\{p_i\}$ such that every subsequence of $\{p_i\}$ contains a subsequence which is r-convergent to p, but $\{p_i\}$ is not r-convergent to p. First, we will show that there exists a sequence $\{V_k\}$ $(k=0,1,2,\ldots)$ of elements of $\boldsymbol{S}$ such that $V_0<V_1<\cdots<V_k<\cdots$ and the set $\{V_k \mid k=0,1,2,\ldots\}$ is cofinal in $\boldsymbol{F}(p)$. Indeed, put $\boldsymbol{S}/\sim=\{S_i \mid i=0,1,2,\ldots\}$ and let U_i be any element of S_i. Put $V_0=U_0$, $1^*=\min\{i \mid U_0, U_1<U_i\}$, $V_1=U_1{}^*$, $2^*=\min\{i \mid U_0,\ldots, U_{i_2}<U_i\}$ and $V_2=U_2{}^*$, where $i_2=\max\{2, 1^*\}$. Generally, for $k=2,3,\ldots$ put $k^*=\min\{i \mid U_0,\ldots, U_{i_k}<U_i\}$ and $V_k=U_k{}^*$, where $i_k=\max\{k, (k-1)^*\}$. Then, by induction, we have a sequence $\{V_k\}$ of elements of $\boldsymbol{S}$ which possesses the desired property. Now, put $V_k=\{v_k(p, m_i{}^k)\}$ $(k, i=0,1,2,\ldots)$. As $p_i \underset{r}{\not\to} p(V_k)$ (i. e. it is false that $p_i \underset{r}{\longrightarrow} p(V_k)$), there exist a member $v_k(p, m_{i_0}{}^k)$ of V_k and a subsequence $\{p_{i_j}{}^k\}$ $(j=0,1,2,\ldots)$ of $\{p_i\}$ such that $p_{i_j}{}^k \notin v_k(p, m_{i_0}{}^k)$ for all j. Since $V_0<V_1$, there exists a member $v_0(p, m_{i_1}{}^0)$ of V_0 such that $v_0(p, m_{i_0}{}^0)\cap v_1(p, m_{i_0}{}^1)\supset v_0(p, m_{i_1}{}^0)$ and $m_{i_0}{}^0<m_{i_1}{}^0$. Assume that for an integer $k\geq 0$ we get the members $v_l(p, m_{i_0}{}^l)$, $v_l(p, m_{i_1}{}^l),\ldots, v_l(p, m_{i_{k-l}}{}^l)$ of V_l $(l=0,\ldots,k)$ such that $v_l(p, m_{i_0}{}^l)\supset v_l(p, m_{i_1}{}^l)\supset\cdots\supset v_l(p, m_{i_{k-1}}{}^l)$, $0\leq m_{i_0}{}^l< m_{i_1}{}^l<\cdots<m_{i_{k-l}}{}^l$ and $v_l(p, m_{i_n}{}^l)\supset v_{l-1}(p, m_{i_{n+1}}{}^{l-1})$, where $n=0,\ldots, l-1$. Since $V_0<\cdots< V_k<V_{k+1}$, there exist the members $v_l(p, m_{i_{k+1-l}}{}^l)$ of V_l, $l=0,\ldots,k+1$, such that $v_l(p, m_{i_{k-l}}{}^l)\supset v_l(p, m_{i_{k+1-l}}{}^l)$, $m_{i_{k-1}}{}^l<m_{i_{k+1-l}}{}^l$ and $v_{k+1}(p, m_{i_0}{}^{k+1})\supset v_k(p, m_{i_1}{}^k)\supset\cdots\supset v_0(p, m_{i_{k+1}}{}^0)$. Therefore, by induction, putting $u_n{}^k=v_k(p, m_{i_n}{}^k)$ $(k, n=0,1,2,\ldots)$ we get the p-f.s.'s $U_k=\{u_n{}^k\}$ $(n=0,1,2,\ldots)$, $k=0,1,2,\ldots$ such that $U_k\sim V_k$ and $u_n{}^{k+1}\supset u_{n+1}{}^k$ for all k, n. Now choose a subsequence $\{q_k\}$ of $\{p_i\}$ such that q_k is a member of $\{p_{i_j}{}^k\}$. Then, $\{q_k\}$ contains no subsequence which is r-convergent to p. This contradicts the supposition.

(b) The "only if" part. Suppose that $\boldsymbol{S}$ is not directed. Then there would exist two $V, U\in \boldsymbol{S}$ which have no common upper element in $\boldsymbol{S}$. Hence, for each $W\in \boldsymbol{S}$ we have $V\not< W$ or $U\not< W$. Put $\boldsymbol{A}=\{W\in \boldsymbol{S} \mid V\not< W\}$ and $\boldsymbol{B}=\{W\in \boldsymbol{S} \mid U\not< W\}$. Since $\boldsymbol{A}\cup \boldsymbol{B}=\boldsymbol{S}$ and $\boldsymbol{S}$ is essentially countable, $\boldsymbol{A}$ and $\boldsymbol{B}$ are essentially countable. We only treat the case that both $\boldsymbol{A}/\sim$ and $\boldsymbol{B}/\sim$ are infinite, because an argument which will be done in this case is applied analogously to other cases. Let $\boldsymbol{A}/\sim= \{A_k \mid k=0,1,2,\ldots\}$, $\boldsymbol{B}/\sim=\{B_k \mid k=0,1,2,\ldots\}$ and W'_k, W^*_k be any elements of A_k, B_k respectively. Put $V=\{v_i\}$, $U=\{u_i\}$, $W'_k=\{w'_i{}^k\}$ and $W^*_k=\{w^*_i{}^k\}$. Since $V\not< W'_k$, there exist a $w'_{i_V}{}^k\in W'_k$ and a sequence $\{p_i{}^k\}$ $(i=0,1,2,\ldots)$ such that $p_i{}^k\in v_i-w'_{i_V}{}^k$ for all i. Similarly, since $U\not< W^*_k$, there exist a $w^*_{i_U}{}^k\in W^*_k$ and a sequence $\{q_i{}^k\}$ $(i= 0,1,2,\ldots)$ such that $q_i{}^k\in u_i-w^*_{i_U}{}^k$ for all i. For $j=0,1,2,\ldots$, put $s_n=p^k{}_j$ and $t_n=q^k{}_j$, where $0\leq k\leq j$ and $n=j(j+1)/2+k$. Then $s_n \underset{r}{\longrightarrow} p(V)$, $t_n \underset{r}{\longrightarrow} p(U)$ and $s_n \underset{r}{\not\to} p(W)$ or $t_n \underset{r}{\not\to} p(W)$ for any $W\in \boldsymbol{S}$, so for any $W\in \boldsymbol{F}(p)$. However, for the sequence $\{x_n\}$, where $x_{2n}=s_n$ and $x_{2n-1}=t_n$ $(n=0, 1, 2,\ldots)$, we have, by the hypothesis, $x_n \xrightarrow[r]{} p$, or there is a $W_0 \in \boldsymbol{F}(p)$ such that $x_n \xrightarrow[r]{} p(W_0)$. This is a contradiction.

COROLLARY 1. *If there exists a $V\in \boldsymbol{F}(p)$ such that $U<V$ for any $U\in \boldsymbol{F}(p)$, then the r-convergence to p is the star convergence.*

DEFINITION. For a subset A of R, denote by $\lambda(A)$ the set $\{p\in R \mid p_i \xrightarrow[r]{} p$ for some sequence $\{p_i\}$ in $A\}$.

COROLLARY 2. *If there exists a $V=\{v_i\}\in \boldsymbol{F}(p)$ such that $p\in R-\lambda(R-v_i)$ $(i=0, 1,2,\ldots)$, then the r-convergence to p is the star convergence.*

COROLLARY 3. *If $\boldsymbol{F}(p)$ is essentially one (i. e. $\boldsymbol{F}(p)/\sim$ consists of only one element), then the r-convergence to p is the star convergence.*

COROLLARY 4. *If every preneighborhood $v(p)$ of p which has a rank is r-open (i. e. $v(p)=R-\lambda(R-v(p))$), then the r-convergence to p is the star convergence.*

From the well-known fact relating the convergence and the star convergence (e. g. see[2]), we immediately have the following proposition.

PROPOPOSITION. *A sequence $\{p_i\}$ is r-star convergent to p if and only if $p_i \xrightarrow[r]{} p$ is identical with that for any subset G of R, $p\in R-\lambda(R-G)$, there is an integer $i'\geq 0$ such that $p_i\in G$ for all $i\geq i'$.*

2. *Examples.*

EXAMPLE 1. Let $E=\{(x,y) \mid x, y$ are real numbers$\}$. Fer a $p\in E$, define preneighborhoods of p as follows: (1) the sets $v(p:\varepsilon, \alpha, \theta)=\{(\delta\cos\varphi, \delta\sin\varphi) \mid 0\leq\delta<\varepsilon, \alpha<\varphi<\alpha+\theta\}$, where ε, α, θ are any real numbers such that $0<\varepsilon$, $0\leq\alpha<2\pi$ and $0<\theta<2\pi$, if $p=(0, 0)$. (2) the sets $v(p:\varepsilon)=\{(x, y) \mid (a-x)^2+(b-y)^2<\varepsilon\}$, where ε is any positive real number, if $p=(a, b)\neq(0, 0)$. Now, define $v(p:\varepsilon, \alpha, \theta)$, $v(p:\varepsilon)$ as the preneighborhoods of rank n $(n=0, 1, 2,\ldots)$ if $[1/\varepsilon]=n$. Then E becomes a ranked space. In this space, by the Theorem, the r-convergence to $p=(0, 0)$ is not the star convergence, because every essentially countable subset of $\boldsymbol{F}(p)$ which is cofinal in $\boldsymbol{F}(p)$ (in fact, such a subset exists at least one) is not directed.

EXAMPLE 2. Let R be a ranked union space of the ranked spaces R_α $(\alpha\in\Sigma)$ (see [3]). For a $p\in R$, let $\boldsymbol{F}_\alpha(p)$ be the totality of p-f. s.'s in R_α. Now assume that $p_i \xrightarrow[r]{} p$ in R implies $p_i \xrightarrow[r]{} p$ in some R_α. Then $p_i \xrightarrow[r]{} p$ in R is equivalent to $p_i \xrightarrow[r]{} p$ in some R_α. Hence, by the Theorem, the r-convergence to $p(\in R)$ in R is the star convergence, if for each $\alpha\in\Sigma$ there exists an essentially countable subset of $\boldsymbol{F}_\alpha(p)$ which is cofinal in $\boldsymbol{F}_\alpha(p)$ and directed.

Appendix

Let R be a *preneighborhoods space*, or a set R such that each point p of R possesses a non-empty family $\mathfrak{N}_p=\{v(p)\}$ of subsets of R, called *preneighborhoods* of p, which satisfies the axiom (A) : (A) $p\in v(p)$ for any $p\in R$ and any $v(p)\in\mathfrak{N}_p$. Then R is said to be a *ranked space* of *indicator* ω_0 (ω_0 is the first transfinite ordinal number), if, to each integer $n\geq 0$, there corresponds a subfamily $\mathfrak{B}_n$ of $\bigcup_{p\in R}\mathfrak{N}_p$ satisfying the axiom(a) : (a) For any $p\in R$, any $u(p)\in\mathfrak{N}_p$, and any integer $m\geq 0$, there exists a $v(p)\in\mathfrak{N}_p$ such that $v(p)\subset u(p)$ and $v(p)\in\mathfrak{B}_n$ for some $n\geq m$. An element which belongs to $\mathfrak{B}_n$ is said to be a *preneighborhood of rank n*. Preneighborhoods of $p\in R$ of rank n are denoted by $v(p, n)$, $u(p, n)$, etc.. For a $p\in R$, a sequence $V=\{v(p, n_i)\}$ $(i=0, 1, 2,\ldots)$, or in short, $\{v_i\}$, of elements of $\mathfrak{N}_p\cap(\bigcup_{n=0}^{\infty}\mathfrak{B}_n)$ is called a *p-fundamental sequence* (abbreviated to *p-f. s.*) if $v(p, n_i)\supset v(p, n_{i+1})$, $n_i\leq n_{i+1}$ and $\sup_i n_i=\omega_0$. Denote by $\boldsymbol{F}(p)$ the totality of p-f.s.'s. For two elements $V=\{v_i\}$, $U=\{u_i\}$ of $\boldsymbol{F}(p)$, the notation $V<U$ means that for any u_i there is a v_j such that $v_j\subset u_i$, further the notation $V\sim U$ means that both $V<U$ and $U<V$. The negation of $V<U$ is written $V\nless U$.

References

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