Remark on the Limit Set of Normal Subgroups of Kleinian Groups

Shigeyasu KAMIYA

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In this paper we shall prove the following result.

THEOREM. *Let G be a kleinian group and H its normal subgroup. Let $\Lambda(G)$ (resp. $\Lambda(H)$) be the limit set of G (resp. H). If $\Lambda(H)$ is not empty, then $\Lambda(H)=\Lambda(G)$.*

1. At first we prepare five lemmas.

LEMMA 1. (Ford [1], Lehner [2]) *If S is a closed set containing at least two points such that $g(S)\subset S$ for all $g\in G$, then $S\supset\Lambda(G)$.*

LEMMA 2. (Ford [1], Lehner [2]) *The limit set is closed and transformed into itself by any transformation of the group.*

LEMMA 3. (Ford [1]) *In the neighborhood of a limit point z there is an infinite number of distinct points congruent to any point of the plane, with, at most, the exception z and of one other point.*

LEMMA 4. *Let G be a kleinian group and H its normal subgroup. Let $F(H)$ be the set consisting of the fixed poiints of H. Then $F(H)$ is G-invariant.*

PROOF. Take z in $F(H)$. Let g be an element with the fixed point z. We shall show that, if $h\in G$, then $h(z)\in F(H)$. We see that hgh^{-1} fixes $h(z)$. Since H is a normal subgroup of G, $hgh^{-1}\in H$. Thus $h(z)$ is contained in $F(H)$.

LEMMA 5. (Lehner [2]) *If a parabolic element and a loxodromic element in G have a common fixed point, then G is not discrete.*

2. Now we are ready to prove our theorem.

PROOF OF THEOREM. It is clear that $\Lambda(G)\supset\Lambda(H)$. It suffices to prove $\Lambda(G)\subset\Lambda(H)$. First we shall show that $\Lambda(H)$ is G-invariant.

Let $z\in\Lambda(H)$. Lemma 3 impies that there is a sequence $\{g_n\}$ of distinct elements in H such that $\lim_{n\to\infty} g_n(x)=z$ for almost all $x\in\hat{C}$. If $g\in G$, then $\lim_{n\to\infty} gg_ng^{-1}(x)=g\ \lim_{n\to\infty} g_n\ (g^{-1}(x))=g\ (z)$. Since $gg_ng^{-1}\in H$, $\lim_{n\to\infty} gg_ng^{-1}(x)\in\Lambda(H)$, so that $g(z)\in\Lambda(H)$. Therefore G leaves $\Lambda(H)$ invariant.

Next we divide into two cases: (1) $\Lambda(H)$ contains at least two points; (2) $\Lambda(H)$ consists of only one point z.

In case (1), we conclude from Lemmas 1 and 2 that $\Lambda(H)=\Lambda(G)$.

Finally we consider the relation between $\Lambda(H)$ and $\Lambda(G)$ in case (2). In this case, H does not contain loxodromic elements and contains a parabolic element with the fixed point z. Since $\Lambda(H)$ is G-invariant, all elements in G must fix z. It follows from Lemma 5 that G contains no loxodromic elements. Therefore G consists of elliptic and parabolic elements with the fixed point z. Hence $\Lambda(G)=\{z\}=\Lambda(H)$.

Thus our theorem is completely proved.

3. In this section we consider the case of $\Lambda(H)=\emptyset$, where H is not trivial. We first note that H is a finite group. It follows from Lemma 4 that $F(H)$ is G-invariant. And it contains at least two points. Lemma 1 implies that $F(H)\subset\Lambda(G)$. Since $F(H)$ is a finite set, the cardinal number of $\Lambda(G)$ is 0, 1, or 2.

If $\Lambda(G)$ is empty, then $\Lambda(H)=\Lambda(G)$.

In the case where $\Lambda(G)$ contains only one point z, $F(H)$ contains at least two points. Since a parabolic element can not leave $F(H)$ invariant, G does not contain parabolic elements. Hence it is impossible that $\Lambda(H)=\{z\}$. Hence $\Lambda(H)\neq\Lambda(G)$.

When $\Lambda(G)$ contains exactly two points, there exists the following example which $\Lambda(H)\neq\Lambda(G)$:

$$G=\{K^nK_1^mz \mid |K|\neq 1,\ K_1=\exp(2\pi i/k),\quad k,\ n,\ m\in Z\}$$

$$H=\{K_1^mz \mid K_1=\exp(2\pi i/k),\quad k,\ m\in Z\}$$

References

[1] L. R. Ford, Automorphic functions, 2nd ed., Chelsea, 1951.

[2] J. Lehner, Discontinuous groups and automorphic functions, Amer. Math. Soc., Math. Surveys, No. 8, Providence, 1964.

Department of Mechanical Science,
Faculty of Science,
Okayama University of Science