On a Certain Combination Theorem

Shigeyasu KAMIYA

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1. Introduction and statement of result

In the study of kleinian groups it is important to construct new kleinian groups. In this paper we shall show a simple combination theorem. Before stating our theorem, we explain some notation and definitions.

We denote the extended complex plane, or Riemann sphere, by $\hat{\mathbf{C}}$. If G is a group of Möbius transformations, and $z \in \hat{\mathbf{C}}$, then z is called a regular point for G ($z \in R(G)$) if there is a neighborhood U of z so that $g(U) \cap U = \emptyset$, for all $g \in G$, $g \neq 1$. If $R(G) \neq \emptyset$, then G is called a kleinian group. A set D is called fundamental set (FS) for G, if

(a) $g(D) \cap D = \emptyset$, for all $g \in G$, $g \neq 1$,

(b) there exist an element g in G and a point w in D such that $g(w) = z$ for any z in $R(G)$.

Our result is the following theorem.

THEOREM. *Let G be a kleinian group. Let $f \notin G$ be an elliptic element with the period 2. Let D_1, D_2 be FS's for G, $\langle f \rangle$, respectively. Let r_1, r_2, r_3 be simple closed curves.*

Assume that

(1) $r_2 \cap r_3$ *is either empty or one point;*

(2) $r_1 \cap r_i$ $(i=2, 3)$ *is only two points;*

(3) $Ext\ r_1 \subsetneqq D_2 \subsetneqq (Int\ r_1)^c$;

(4) *there exists g in G such that* $g(r_2) = r_3$;

(5) $f(r_2) = r_3$;

(6) $(Int\ r_1) \setminus [(\overline{Int\ r_1}) \cap (Int\ r_2)] \setminus [(\overline{Int\ r_1}) \cap (Int\ r_3)] \subsetneqq D_1 \subsetneqq (Ext\ r_2) \cap (Ext\ r_3)$;

(7) $D_1 \cap D_2 \neq \emptyset$;

Then

(I) *Γ, the group generated by G and f, is kleinian;*

(II) $D=D_1 \cap D_2$ *is FS for* Γ.

2. Proof of theorem

First we shall show that two points in $D=D_1 \cap D_2$ are not equivalent under Γ. Let z be a point in D. We divide into five cases.

i) Since $z \in D_1$, $g(z) \in D_1^c$ for any $g \in G \setminus \{1\}$. Hence $g(D) \cap D = \emptyset$ for any $g \in G \setminus \{1\}$.

ii) As $z \in D_2$, $f(z) \in D_2^c$. Therefore $f(D) \cap D = \emptyset$.

iii) We consider an element which is of the form hf, where $h \in G$. Since $f(z) \in D_2^c$, $f(z) \in Int\ r_1$. If $f(z) \in (Int\ r_1) \cap (Int\ r_2)$, then $z \in (Int\ r_3) \cap (Ext\ r_1)$. Hence $z \notin D_1 \cap D_2$. This contradicts our assumption that z is a point in D. Therefore $f(z) \notin (Int\ r_1) \cap (Int\ r_2)$. If $f(z) \in (Int\ r_1) \cap (Ext\ r_2) \cap (Ext\ r_3)$, then $hf(z) \in D_1^c$. Hence $hf(D) \cap D = \emptyset$. If $f(z) \in D$, then $z \in (Int\ r_2) \cap (Ext\ r_1)$. Hence $z \notin D_1 \cap D_2$. This is a contrdiction. We conclude that $hf(D) \cap D = \emptyset$.

iv) In this case, we show that $fh(D) \cap D = \emptyset$ for every h in G. If $h(z) \in (Int\ r_2) \cup (Int\ r_3)$, then $fh(z) \in (Int\ r_2) \cup (Int\ r_3)$. Hence $fh(z) \in (D_1 \cap D_2)^c$. Therefore $fh(D) \cap D = \emptyset$. If $h(z) \in [(Int\ r_2) \cup (Int\ r_3)]^c$, then $fh(z) \in D_2^c$. Hence $fh(D) \cap D = \emptyset$.

v) In general case, iii) and iv) leads to our conclusion.

Next we shall prove that any point in $R(\Gamma)$ is Γ-equivalent to some point in D. It is clear that $R(G) \supset R(\Gamma)$. It remains to show that a point in $(Int\ r_1) \cap (Ext\ r_2) \cap (Ext\ r_3)$ is Γ-equivalent to some point in D. Let z be in $(Int\ r_1) \cap (Ext\ r_2) \cap (Ext\ r_3)$. When $f(z) \in D$, we have nothing to prove. If $f(z) \notin D$, then $f(z) \in D_2 \setminus D \setminus [(Int\ r_2) \cap (Ext\ r_1)] \setminus [(Int\ r_3) \cap (Ext\ r_1)]$. There exist an element h in G and a point x in D_1 such that $h(x)=f(z)$. It is easy to show that $gf(z) \in (Int\ r_3)$. Since $(Int\ r_3)$ is not contained in D_1, there are an element k and a point y in D_1 such that $k(y)=gf(z)$. It follows that $gf(z)=gh(x)=k(y)$. Now we note that $x \neq y$. Therefore x and y are G-equivalent in D_1. This is a contradiction. Thus $f(z)$ is contained in D. Q. E. D.

3. Application

Using our theorem, we can construct a fuchsian group of the first kind from a fuchsian group of the second kind.

Let G be a fuchsian group of the second kind with one funnel. Let g be a hyperbolic element in G corresponding to the funnel. Let $f \notin G$ be an elliptic element with the period 2 whose isometric circle is an axis of g. We denote a Ford polygon for G (resp. $\langle f \rangle$) by R_1 (resp. R_2). Then Γ, the group by G and f, is a fuchsian group of the first kind and $R_1 \cap R_2$ is a Ford polygon for Γ.

reference

[1] L. R. Ford, Automorphic functions, 2nd ed., Chelsea, 1951.

Department of Mechanical Science,
Faculty of Science,
Okayama Uuniversity of Science