A note on metrizability of ranked spaces

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1. Let R be a topological space with the depth $\omega(R)$ and ω be an inaccessible ordinal number such that $\omega_0 \leq \omega \leq \omega(R)$. The space R is called a *topological ranked space* if, for each ordinal number α, $0 \leq \alpha < \omega$, there corresponds a family of neighborhoods of points of R, $\mathfrak{V}_\alpha$, which satisfies the following axiom (a) :

(a) For any neighborhood $v(p)$ of $p \in R$ and any ordinal number α, $0 \leq \alpha < \omega$, there exist an ordinal number β and a neighborhood $u(p)$ of p such that $\alpha \leq \beta < \omega$, $u(p) \subset v(p)$ and $u(p) \in \mathfrak{V}_\beta$.

The number ω is called the *indicator* of the topological ranked space R. Members which belong to $\mathfrak{V}_\alpha$ are named neighborhoods of rank α. (For the notion of the depth of spaces and the definition of ranked spaces, see [1])

For $p \in R$, we denote by $\mathfrak{V}_\alpha(p)$ the set of all neighborhoods of p of rank α.

2. *Definition of the fundamental sequence of neighborhoods.* Let R be a topological ranked space with the indicator ω and p be a point of R. A sequence $V(p) = \{v_\alpha(p)\}$ $(0 \leq \alpha < \omega)$ of neighborhoods of p is said to be *fundamental* if the following conditions are fulfilled :

(1) $v_0(p) \supset v_1(p) \supset \cdots \supset v_\alpha(p) \supset \cdots$, $0 \leq \alpha < \omega$.

(2) There exists a monotone increasing sequence of ordinal numbers : $\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_\alpha \leq \cdots$, $0 \leq \gamma_\alpha < \omega$ such that $v_\alpha(p) \in \mathfrak{V}_{\gamma_\alpha}(p)$ for α, $0 \leq \alpha < \omega$, and $\sup_\alpha \gamma_\alpha = \omega$.

For $p \in R$, we denote by $\mathbf{F}(p)$ the totality of fundamental sequences of neighborhoods of p. Let $V(p) = \{v_\alpha(p)\}$ and $U(p) = \{u_\alpha(p)\}$ be elements of $\mathbf{F}(p)$. If, for any $v_\alpha(p)$, there exists a $u_\beta(p)$ such that $v_\alpha(p) \supset u_\beta(p)$, then we write this by $U(p) < V(p)$ or $V(p) > U(p)$. Furthermore, if $U(p) < V(p)$ and $V(p) < U(p)$, then we say that $V(p)$ and $U(p)$ are equivalent and write $V(p) \sim U(p)$.

Clearly, for each $p \in R$, "$\sim$" is the equivalence relation on elements of $\mathbf{F}(p)$ and, by identifying equivalent elements, $\mathbf{F}(p)$ becomes a partially ordered set.

Proposition 1. *For each $p \in R$, $\mathbf{F}(p)$ is directed downward, i.e., for any two elements $V(p)$, $U(p) \in \mathbf{F}(p)$, there exists a $W(p) \in \mathbf{F}(p)$ such that $W(p) < V(p)$ and $W(p) < U(p)$.*

This follows easily from (a) and the definition of the fundamental sequence.

3. A metrizable topological space is a topological ranked space with the depth ω_0 and the indicator ω_0.

Definition. Let E be a set and A, B be families of subsets of E. If for any $a \in A$ there exists a $b \in B$ such that $a \supset b$, and *vice versa*, then we say that A and B are equivalent.

Definition. A topological ranked space R with the indicator ω is said to be metrizable if there exirts a metric d on R such that, for each $p \in R$, the family of subsets of R, $\mathfrak{B}(p) = \{B(p:\varepsilon) \mid 0 < \varepsilon < +\infty\}$, where $B(p:\varepsilon) = \{q \in R \mid d(p, q) < \varepsilon\}$, is equivalent to $\bigcup_{\alpha<\omega} \mathfrak{B}_\alpha(p)$.

From this definition it is obvious that if a topological ranked space is metrizable, then its indicator is ω_0. On the metrizability problem of spaces, the following Frink's Theorem ([3]) is well-known:

Theorem. *A topological space S is metrizable if and only if, for each $p \in S$, there exists a sequence of neighborhoods of p, $\{v_n(p)\}$ $(n=1, 2, \cdots)$, which satisfies*

(1) $v_1(p) \supset v_2(p) \supset \cdots \supset v_n(p) \supset \cdots$,

(2) $\bigcap_n v_n(p) = \{p\}$,

(3) $\{v_n(p)\}$ *is equivalent to the neighborhoods system of p,*

and

(4) *For any $v_n(p)$, there exists a natural number $m = (p, n)$ such that if q is any point of S for which $v_m(p)$ and $v_m(q)$ have a point in common, then $v_m(q) \subset v_n(p)$.*

By putting $\mathfrak{B}_n = \{v_{n+1}(p) \mid p \in S\}$ $(n=0, 1, 2, \cdots)$, a topological space S which fulfills the above conditions (1), (3) becomes a topological ranked space with the indicator ω_0. Under what conditions, is a topological ranked space metrizable? In the next section, we shall study Frink's Theorem type metrizability of topological ranked spaces.

4. Let R be a topological ranked space with the indicator ω.

Definition. Let p be a point of R and $\mathbf{T}(p)$ be a subset of $\mathbf{F}(p)$. An element $V(p) \in \mathbf{T}(p)$ is called a minimum element of $\mathbf{T}(p)$ if $V(p) < U(p)$ for any $U(p) \in \mathbf{T}(p)$.

Proposition 2. *R is metrizable if the following conditions are fulfilled:*

1° $\omega = \omega_0$.

2° *For each $p \in R$, there exists a minimum element of $\mathbf{F}(p)$, $V(p) = \{v_n(p)\}$, which satisfies the condition (4) in Frink's Theorem.*

3° *For each $p \in R$, there exists a $U(p) = \{u_n(p)\} \in \mathbf{F}(p)$ such that $\bigcap_n u_n(p) = \{p\}$.*

Proof. Let p be any point of R. As $V(p)$ is a minimum element of $\mathbf{F}(p)$ for any $u_n(p)$ there exists a $v_m(p)$ such that $u_n(p) \supset v_m(p)$. Hence $\{p\} = \bigcap_n u_n(p) \supset \bigcap_n v_n(p) \ni p$ and so $\bigcap_n v_n(p) = \{p\}$. Let $w_0(p)$ be any neighborhood of p of rank γ_0. Then, by (a), there exist a neighborhood $w_1(p)$ of p and a natural number γ_1, $\gamma_0 < \gamma_1 < \omega_0$, such that $w_0(p) \supset w_1(p)$ and $w_1(p) \in \mathfrak{B}_{\gamma_1}$. By successive applications of (a), we obtain a fundamental sequence of neighborhoods of p, $W(p) = \{w_n(p)\}$. Since $W(p) \in \mathbf{F}(p)$ and $V(p)$ is a minimum element of $\mathbf{F}(p)$, we have $V(p) < W(p)$. Thus there exists a neighborhood $v_n(p)$ such that $w_0(p) \supset v_n(p)$. This proves that $\{v_n(p)\}$ is equivalent to $\bigcup_n \mathfrak{B}_n(p)$, so that by (a) $\{v_n(p)\}$ is equivalent to the neighborhoods system of p. Hence, by Prooosition 3 and Frink's Theorem, R is metrizable.

Proposition 3. *R is metrizable if and omly if R is metrizable when it is considered as a topological space.*

This follows at once from (a) and the definition of metrizability of topological ranked spaces.

Proposition 4. *In Proposition 2, the conditions 1°, 2° and 3° are necessary for which R is mrtrizable.*

This will be proved, by an application of (a), in the same manner stated in [3] pp. 141–142.. So the detailed proof is omitted.

For each $p \in R$, we can define the *depth of* $\mathbf{F}(p)$ in the same manner of defining the depth of R at p (see [1]). We denote by $\omega(\mathbf{F}(p))$ the depth of $\mathbf{F}(p)$.

By virtue of Proposition 1, $\omega(\mathbf{F}(p))$ becomes an inaccessible ordinal number.

Hereafter, for $p \in R$ and a neighborhood $v(p) \in \bigcup_\alpha \mathfrak{B}_\alpha(p)$, we denote by $r(v(p))$ the rank of $v(p)$.

Proposition 5. *For $p \in R$, we have $\omega \leq \omega(\mathbf{F}(p))$.*

First we have a next lemma.

Lemma. *For $V(p)$, $U(p) \in \mathbf{F}(p)$, if $V(p) > U(p)$, then there exists a $W(p) \in \mathbf{F}(p)$ such that $U(p) \sim W(p)$, $v_\alpha(p) \supset w_\alpha(p)$* and $r(v_\alpha(p)) < r(w_\alpha(p))$ for all α, $0 \leq \alpha < \omega$, where $V(p) = \{v_\alpha(p)\}$ and $W(p) = \{w_\alpha(p)\}$.

This follows easily from the definition of the fundamental sequence and the relation $V(p) > U(p)$.

Proof of Proposition 5. Suppose $\omega > \omega(\mathbf{F}(p))$. From the definition of the depth of $\mathbf{F}(p)$, there exists a monotone decreasing sequence of elements of $\mathbf{F}(p)$:

$$V_0(p) > V_1(p) > \cdots > V_\beta(p) > \cdots, \ 0 \leq \beta < \omega(\mathbf{F}(p))$$

such that, for no $U(p) \in \mathbf{F}(p)$, we have $V_\beta(p) > U(p)$ for all β, $0 \leq \beta < \omega(\mathbf{F}(p))$. Put $V_\beta(p) = \{v_\alpha{}^\beta(p)\}$ $(0 \leq \alpha << \omega)$, $0 \leq \beta < \omega(\mathbf{F}(p))$. Again put $V_0(p) = W_0(p)$ and

$v_\alpha^0(p)=w_\alpha^0(p)$, $0\leq\alpha<\omega$. Now assume that, for an ordinal number β, $0\leq\beta<\omega(\mathbf{F}(p))$, we have elements $W_\xi(p)=\{w_\alpha^\xi(p)\}$ of $\mathbf{F}(p)$, $0\leq\xi<\beta$ such that

(1) $V_\xi(p)\sim W_\xi(p)$, $0\leq\xi<\beta$

and

(2) if $0\leq\xi<\eta<\beta$, then $w_\alpha^\xi(p)\supset w_\alpha^\eta(p)$ for all α, $0\leq\alpha<\omega$.

We consider two cases.

Case 1. β is a limiting number. Assume that, for an ordinal number α, $0\leq\alpha<\omega$, we have a sequence of members of of $V_\beta(p)$:

$w_0^\beta(p)\supset w_1^\beta(p)\supset\cdots\supset w_\delta^\beta(p)\supset\cdots$, $0\leq\delta<\alpha$

such that $0\leq r(w_0^\beta(p))<r(w_1^\beta(p))<\cdots<r(w_\delta^\beta(p))<\cdots<\omega$ and $w_\delta^\beta(p)\subset\bigcap_{\xi<\beta}w_\delta^\xi(p)$ for all δ, $0\leq\delta<\alpha$, Since $W_\xi(p)>V_\beta(p)$ for all ξ, $0\leq\xi<\beta$ and $w_\alpha^0(p)\supset w_\alpha^1(p)\supset\cdots\supset w_\alpha^\xi(p)\supset\cdots$, $0\leq\xi<\beta$, there exists a sequence of members of $V_\beta(p)$:

$v_{\gamma_0}^\beta(p)\supset v_{\gamma_1}^\beta(p)\supset\cdots\supset v_{\gamma_\xi}^\beta(p)\cdots$, $0\leq\xi<\beta$, $0\leq\gamma_\xi<\omega$

such that $w_\alpha^\xi(p)\supset v_{\gamma_\xi}^\beta(p)$ for all ξ, $0\leq\xi<\beta$. Put $w_\alpha^\beta(p)=v_\gamma^\beta(p)$, where $v_\gamma^\beta(p)$ is such that $r(v_\gamma^\beta(p))=\min\{r(v_\zeta^\beta(p))\mid \sup_{\delta<\alpha}r(w_\delta^\beta(p)),\ \sup_{\xi<\beta}r(w_{\gamma_\xi}^\beta(p))<r(v_\zeta^\beta(p))<\omega\}$. The number $r(v_\gamma^\beta(p))$ really exists, since α, $\beta<\omega$ and ω is inaccessible. Clearly we have $w_\alpha^\beta(p)\subset\bigcap_{\xi<\beta}v_{\gamma_\xi}^\beta(p)\subset\bigcap_{\xi<\beta}w_\alpha^\xi(p)$. Thus, by the transfinite induction, we obtain an element $W_\beta(p)=\{w_\alpha^\beta(p)\}\in\mathbf{F}(p)$ such that

(3) $V_\beta(p)\sim W_\beta(p)$

and

(4) if $0\leq\xi<\beta$, then $w_\alpha^\xi(p)\supset w_\alpha^\beta(p)$ for all α, $0\leq\alpha\leq\omega$.

Case 2. β is an isolated number. Since $W_{\beta-1}(p)>V_\beta(p)$, by the above Lemma, there exists an element $W_\beta(p)=\{w_\alpha^\beta(p)\}\in\mathbf{F}(p)$ which satisfies (3) and (4).

Therefore, by the trantfinite induction, we obtain elements $W_\beta(p)=\{w_\alpha^\beta(p)\}\in\mathbf{F}(p)$, $0\leq\beta<\omega(\mathbf{F}(p))$, such that

(5) $V_\beta(p)\sim W_\beta(p)$, $0\leq\beta<\omega(\mathbf{F}(p))$

and

(6) if $0\leq\xi<\eta<\omega(\mathbf{F}(p))$, then $w_\alpha^\xi(p)\supset w_\alpha^\eta(p)$ for all α, $0\leq\alpha<\omega$.

From this and the assumption $\omega(\mathbf{F}(p))<\omega$, using an analogous argument stated in case 1, we can find an element $U(p)\in\mathbf{F}(p)$ such that $W_\beta(p)>U(p)$ for every β, $0\leq\beta<\omega(\mathbf{F}(p))$. This is a contradiction.

Definition. Let p be a point of R and $\mathbf{T}(p)$ be a subset of $\mathbf{F}(p)$. An element $V(p)\in\mathbf{T}(p)$ is called a minimal element of $\mathbf{T}(p)$ if $V(p)>U(p)$, $U(p)\in\mathbf{T}(p)$ yield $V(p)\sim U(p)$.

Proposition 6. *Let p be a point of R. If there exists a non-empty subset* $\mathbf{T}(p)\subset$

$\mathbf{F}(p)$ *such that*

1) *for any* $V(p) \in \mathbf{F}(p)$ *there exists a* $U(p) \in \mathbf{T}(p)$ *which satisfies* $V(p) > U(p)$ *and*

2) $\sup \{\gamma \mid \gamma$ *is a type of a maximal sequence of elements of* $\mathbf{T}(p)\} \leq \omega$,

then there exists a minimal element of $\mathbf{T}(p)$.

Proof. Suppose that the proposition does not hold. Let $V(p)$ be any element of $\mathbf{T}(p)$. As there is no maximal element in $\mathbf{T}(p)$, we have a sequence of elements of $\mathbf{T}(p)$:

(7) $V(p) = V_0(p) > V_1(p) > \cdots > V(p) > \cdots$. $0 \leq \beta < \gamma$ (γ is an ordinal number)

such that, for no $U(p) \in \mathbf{T}(p)$, we have $V_\beta(p) > U(p)$ for all β, $0 \leq \beta < \gamma$. Since (7) becomes a maximal sequence in $\mathbf{T}(p)$, $\gamma \leq \omega$. But, by 1). (7) is also a maximal sequece in $\mathbf{F}(p)$, hence $\omega(\mathbf{F}(p)) \leq \gamma$. Thus by Proposition 5 we have $\omega(\mathbf{F}(p)) = \gamma = \omega$. An analogous argument stated in the proof of Proposition 5 provides elements $W_\beta(p) = \{w_\alpha^\beta(p)\}$ of $\mathbf{F}(p)$, $0 \leq \beta < \omega$, such that

(8) $V_\beta(p) \sim W_\beta(p)$, $0 \leq \beta < \omega$

and

(9) if $0 \leq \xi < \eta < \omega$, then $w_\alpha^\xi(p) \supset w_\alpha^\eta(p)$ and $r(w_\alpha^\xi(p)) < r(w_\alpha^\eta(p)$ for all α, $0 \leq \alpha < \omega$.

Put, for each α, $0 \leq \alpha < \omega$, $w_\alpha(p) = w_\alpha^\alpha(p)$. Then $W(p) = \{w_\alpha(p)\}$ is an element of $\mathbf{F}(p)$ such that $V_\beta(p) > W(p)$ for all β, $0 \leq \beta < \omega = \gamma$. But, by 1), there exists an element $U(p) \in \mathbf{T}(p)$ such that $W(p) > U(p)$. This contradicts the maximality of (7).

Corollary 1. *Let* $p \in R$. *If there exists a non-empty subset* $\mathbf{T}(p) \subset \mathbf{F}(p)$ *which satisfies conditions 1), 2) in Proposition 6, then there exists a minimal element of* $\mathbf{F}(p)$.

Corollary 2. *Let* $p \in R$. *If there exists a totally ordered non-empty subset* $\mathbf{T}(p) \subset \mathbf{F}(p)$ *which satisfies conditions 1), 2) in Proposition 6, then there exists a minimum element of* $\mathbf{F}(p)$.

Proposition 7. *Let* $p \in R$. *If* $\mathbf{F}(p)$ *is finite by identifying equivalent elements, then there exists a minimum element of* $\mathbf{F}(p)$.

This follows at once from Proposition 1.

Definition. A mapping $\tau: R \to \bigcup_{p \in R} \mathbf{F}(p)$ is called a f-map if $\tau(p) \in \mathbf{F}(p)$ for each $p \in R$. For f-maps τ, τ', we write $\tau' \geq \tau$ if $\tau(p) > \tau'(p)$ for each $p \in R$.

Proposition 8. *R is metrizable if, for each* $p \in R$, $\mathbf{F}(p)$ *is countable by means of identifying equivalent elements and the following conditions are fulfilled:*

1′ *The condition 1° in Proposition 2.*

2′ *There exists a f-map τ such that, for any f-map $\tau' \geq \tau$, the condition (4) in Frink's Theorem holds for elements of $\tau'(R)$.*

3′ *The condition 3° in Proposition 2.*

Proof. From Proposition 2, it is sufficient to show that there exists a minimum element of $\mathbf{F}(p)$ for each $p \in R$. Let p be a point of R. For $V(p) \in \mathbf{F}(p)$, put $[V(p)] = \{U(p) \in \mathbf{F}(p) \mid V(p) \sim U(p)\}$ and put $\mathbf{E}(p) = \{[V(p)] \mid V(p) \in \mathbf{F}(p)\}$. If $\mathbf{E}(p)$ is finite, from Prosition 7, there is nothing to prove. If $\mathbf{E}(p)$ is not finite, we can write $\mathbf{E}(p) = \{C_0, C_1, C_2, \cdots\}$. Let $V_n(p)$ be any element of C_n, $0 \leq n < \omega_0$. From Proposition 1 we obtain, by induction, a sequence of elements of $\mathbf{F}(p)$:

$$W_0(p) > W_1(p) > \cdots > W_n(p) > \cdots, \ 0 \leq n < \omega_0$$

such that $V_n(p) > W_n(p)$ for all n. Clearly the set $\{W_0(p), W_1(p), W_2(p), \cdots\}$ is totally ordred and satisfies conditions 1), 2) in Proposition 6. Hence, by Corollary 2, there exists a minimum element of $\mathbf{F}(p)$.

5. *Notes.*

(1) Clearly, the condition 3° in Proposition 2 can be replaced with the following axiom (D^*) in [2]:

(D^*) For any distinct points $p, q \in R$, there exists a non-negative integer $m = m(p, q)$ such that $v(p) \cap v(q) = 0$ for any $v(p), v(q) \in \bigcup_{m \leq n < \omega_0} \mathfrak{V}_n$.

(2) In Proposition 8, can we replace the condition 2′ with the condition below?

For any $v_n(p) \in \mathfrak{V}_n$ there exist a non-negative integer $m = m(v_n(p))$ and $v_m(p) \in \mathfrak{V}_m(p)$ such that $n \leq m < \omega_0$ and $v_n(p) \supset v_m(p)$, and if $v_m(p)$ has a point in common with some $v_m(q) \in \mathfrak{V}_m(q)$ then there exists $u_m(q) \in \mathfrak{V}_m(q)$ which satisfies $u_m(q) \subset v_n(p)$ and $u_m(q) \cap v_m(p) \neq 0$.

Further, in the above condition, if we change under "if $\cdots$" to "if $v_m(q)$ is any neighborhood of rank m for which $v_m(p) \cap v_m(q) \neq 0$, then $v_m(q) \subset v_n(p)$", how goes the above question?

(3) In Proposition 7 and Proposition 8, $\mathbf{F}(p)$ need not be finite or countable by means of identifying equivalent elements. It is enough to assume an existence of a non-empty subset $\mathbf{T}(p) \subset \mathbf{F}(p)$ which is finite (in Proposition 7) or countable (in Proposition 8) by means of identifying equivalent elements and satisfies the condition 1) in Proposition 6.

References

[1] K. Kunugi: *Sur la méthode des espaces rangés* I, Proc. Japan Acad. 42 (1966) pp. 318-322.

[2] K. Kunugi: *Sur la méthode des espaces rangés* II, Proc. Japan Acad. 42 (1966) pp. 549-554.

[3] A.H. Frink: *Distance functions and the metrization problem*, Bull. Math. Soc. 43 (1937) pp. 133-142.