A NOTE ON THE DEPTH OF SPACES

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Let R be a space with neighbourhoods, or a non-empty set R such that each point p of R possesses a non-empty family consisting of subsets $V(p)$ of R called neighbourhoods of p. For such space R, Prof. Kunugi defined the notion "depth" of R and remarked that the depth $\omega(R)$ of R became an inaccessible Cantor's ordinal number as long as R satisfied Hausdorff's axiom (B):

(B) For any point p of R and any two neighbourhoods $U(p)$, $V(p)$ of p, there exists a neighbourhood $W(p)$ of p such that $W(p) \subset U(p) \cap V(p)$. (see 1).)

Here, we give a direct proof of this fact and some examples on the depth of spaces.

1. Proof of inaccessibility of $\omega(R)$:

(An ordinal number γ is called inaccessible, if it is a limiting number and for any ordinal number β, $\beta<\gamma$, we always have $\sup_{0 \leq \xi < \beta} \alpha(\xi) < \gamma$, where $\alpha(\xi)$ is any function defined for ξ, $0 \leq \xi < \beta$ and $0 \leq \alpha(\xi) < \gamma$)

As $\omega(R) = \inf_{p \in R} \omega(p) = \min_{p \in R} \omega(p)$, where $\omega(p)$ is the depth of R at $p \in R$, it is sufficient to show that for each point p of R, $\omega(p)$ is an inaccessible ordinal number. Let p be any point of R. By the definition of $\omega(p)$, $\omega(p)$ is the smallest of ordinal numbers γ such that there exists a monotone decreasing sequence of neighbourhoods of p:

$$V_0(p) \subset V_1(p) \subset \cdots V_\xi(p) \subset \cdots, \quad 0 \leq \xi < \gamma$$

and

$$\bigcap_{0 \leq \xi < \gamma} V_\xi(p) \supset U(p) \text{ for no neighbourhood } U(p) \text{ of } p.$$

Hence we have a sequence of neighbourhoods of p:

$$U_0(p) \supset U_1(p) \supset \cdots \supset U_\xi(p) \supset \cdots, \quad 0 \leq \xi < \omega(p)$$

and

$$\bigcap_{0 \leq \xi < \omega(p)} U_\xi(p) \supset U(p) \text{ for no neighbourhood } U(p) \text{ of } p.$$

Since R satisfies Hausdorff's axiom (B), it is clear that $\omega(p)$ is a limiting number.

Suppose $\omega(p)$ is not inaccessible. Then there would exist an ordinal number β, $0<\beta<\omega(p)$ and a function $\alpha(\xi)$ defined for ξ, $0\leq\xi<\beta$ and $0\leq\alpha(\xi)<\omega(p)$ such that $\sup_{0\leq\xi<\beta} \alpha(\xi)=\omega(p)$. Let $W(\beta)=\{\alpha|\ \alpha \text{ is ordinal and } 0\leq\alpha<\beta\}$ and $E=\{\eta\in W(\beta)|\ \alpha(\xi)\leq\alpha(\eta) \text{ for any } \xi\leq\eta\}$. Clearly, for any η, $\eta'\in E$, $\eta\leq\eta'$ implies $\alpha(\eta)\leq\alpha(\eta')$. Furtheremore we have $\sup_{\eta\in E} \alpha\ (\eta)=\omega(p)$. Otherwise, there would exist an ordinal number ζ, $0\leq\zeta<\beta$ such that $\alpha(\eta)<\alpha(\zeta)$ for any $\eta\in E$. Let ζ^* be the smallest of such ζ's. As $0\leq\zeta^*<\beta$, either $\zeta^*\in E$ or $\zeta^*\in W(\beta)-E$. In the former case, we have $\alpha(\zeta^*)<\alpha(\zeta^*)$. In the latter case, there exists ξ such that $\xi<\zeta^*$ and $\alpha(\zeta^*)<\alpha(\xi)$, hence ξ satisfies $\xi<\zeta^*$ and $\alpha(\eta)<\alpha(\xi)$ for any $\eta\in E$. Afterall, in both cases we have contradictions. Since $O\neq E\subset W(\beta)$ and $W(\beta)$ is well-ordered, there exists an ordinal number β^*, $0<\beta^*\leq\beta$ and E is similar to $W(\beta^*)=\{\xi|\xi \text{ is ordinal and } 0\leq\xi<\beta^*\}$ i.e. there exists a bijection φ of $W(\beta^*)$ onto E such that $\varphi(\xi)\leq(\xi')$ for any ξ, ξ', $0\leq\xi\leq\xi'<\beta^*$. So, if we put $U_\xi^*(p)=U_{\alpha(\varphi(\xi))}(p)$, $0\leq\xi<\beta^*$, we have the monotone decreasing sequence of neighbourhoods of p:

$$U_0^*(p)\subset U_1^*(p)\supset\cdots\supset U_\xi^*(p)\supset\cdots,\ 0\leq\xi<\beta^*$$

which satisfies

$$\bigcap_{0\leq\xi<\beta^*} U_\xi^*(p) \supset U(p) \text{ for no neighbourhood } U(p) \text{ of } p.$$

This contradicts the definition of $\omega(p)$, since $0<\beta^*\leq\beta<\omega(p)$.

Q.E.D.

2. Examples:

(1) Any metrizable topological space has the depth ω_0, where ω_0 is the first transfinite ordinal number.

(2) Consider the space D which appears in the L. Schwarz's distridution theory. i.e. D is the totality of the real valued function $f=f(x_1,\cdots,x_n)$ which is defined on n-dimensional Euclidean space, has compact support and is infinitely differentiable with respect to each x_i. L. Schwarz defined the (locally convex) topology on D. (3).)

Although D is not metrizable, the depth of D is ω_0. (2).)

(3) Let γ be any inaccessible ordinal number larger than ω_0. Then there exists a space S having a neighbourhoods system $\{U_\xi(p)|0\leq\xi<\gamma\}$ for each $p\in S$ which satisfies the following conditions:

i) $\bigcap_{0\leq\xi<\gamma} U_\xi(p)\ni p$.

ii) If $0 \leq \xi < \eta < \gamma$ then $U_\xi(p) \supset U_\eta(p)$.

iii) If $0 \leq \eta < \gamma$ then $\bigcap_{0 \leq \xi < \eta} U_{\alpha(\xi)}(p_\xi)$ is an open set of S, where $\alpha(\xi)$ is any function defined for ξ, $0 \leq \xi < \eta$ and $0 \leq \alpha(\xi) < \gamma$.

(A subset G of S is open if for each $p \in G$ there is a $U_\xi(p)$ such that $U_\xi(p) \subset G$)

In fact, we can construct concretely such a space. (For example, see 4). pp 69–74.)

Clearly, the space S has the depth $\gamma (> \omega_0)$.

(4) Let μ be an isolated ordinal number. Then Sikorski's ω_μ-additive spaces have the depth ω_μ. (5).)

(ω_μ is an initial ordinal number such that the set of all ordinals ξ, $0 \leq \xi < \omega_\mu$ is of the power $\aleph_\mu$)

Note :

The notion of the depth of spaces was introduced by giving a new method in mathematical analysis——so called "the method of ranked spaces".

The main purpose of the method of ranked spaces is to embed the gap between the method of metric spaces and that of general topological spaces, or a generalization of the method of metric spaces. In mathematical analysis there are many important spaces with the depth ω_0 which are not metrizable. In fact, the spaces with the depth ω_0 (detailedly say, with the "indicator" ω_0 (1).)) are considered to be generalized case of the method of metric spaces.

References

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