DRIVING-POINT IMMITTANCE FUNCTIONS OF GENERALIZED EXPONENTIAL LINE SECTIONS

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ABSTRACT——In a nonuniform pseudo-distortionless transmission line of finite length, the locus on the s-plane of the poles and zeros of the driving-point immittance function depends on the primary line constants, and the critical frequencies on the pole-zero distribution depend on the shape of the distributed series inductance and shunt capacitance. This paper elucidates the general properties of the pole-zero distribution of the driving-point immittance function for a nonuniform pseudo-distortionless transmission line of finite length, and then shows the pole-zero distribution for the generalized exponential line section.

1. INTRODUCTION

We consider a two-port section composed of only a nonuniform pseudo-distortionless transmission line of finite length. The pole-zero distribution of the driving-point immittance function of the section can be obtained by the aid of classical Sturm-Liouville theory. The locus of the poles and zeros on the s-plane depends on the primary line constants, and the distribution of the critical frequencies is established by the corresponding eigenvalue. The deflection of the eigenvalues depends on the shape of the distributed series inductance and shunt capacitance.

Except for the case of homogeneous media, generally the propagation constant of a transmission line is nonuniform along the line. Thus, in order to simplify the model in question, we adopt the coordinate system with electrically modified length instead of one with physical length as a spatial coordinate system along the line.

2. GENERAL PROPERTIES

We assume a nonuniform pseudo-distortionless transmission line of length l, of which the distributed series inductance and shunt capacitance per unit length at the position x are $L(x)$ and $C(x)$, respectively, and are at least twice continuously differentiable with respect to x. On physical grounds both $L(x)$ and $C(x)$ are finite and positive, thus the function,

$$\bar{x}(x)=\int_0^x \sqrt{L(\xi)C(\xi)}d\xi \Big/ \int_0^l \sqrt{L(\xi)C(\xi)}d\xi , \tag{1}$$

is a monotonously increasing function of x in a narrow sense. Namely, the function in (1) is a single-valued function of x, and x has one to one correspondence to $\bar{x}$, and especially

$$\bar{x}(0)=0 \quad \text{and} \quad \bar{x}(l)=1$$

Defining the electrically modified length along the transmission line in terms of (1), the distributed series inductance and shunt capacitance per unit length of $\bar{x}$ are given by

$$\bar{L}(\bar{x})=\sqrt{L(x)/C(x)}\int_0^l \sqrt{L(x)C(x)}dx$$

and

$$\bar{C}(\bar{x})=\sqrt{C(x)/L(x)}\int_0^l \sqrt{L(x)C(x)}dx , \tag{2}$$

respectively.

A pair of line equations in transform notation for the nonuniform pseudo-distortionless transmission line can be written in terms of

$$\begin{aligned} \bar{V}_{\bar{x}}'(\bar{x}, s) &= -(s+r)\bar{L}(\bar{x})\bar{I}(\bar{x}, s) \\ \bar{I}_{\bar{x}}'(\bar{x}, s) &= -(s+g)\bar{C}(\bar{x})\bar{V}(\bar{x}, s) \end{aligned} \tag{3}$$

where s is the complex frequency variable ($s=\sigma+j\omega$), both of r and g are finite and nonnegative constants characterizing the pseudo-distortionless line, and $\bar{V}(\bar{x}, s)$ and $\bar{I}(\bar{x}, s)$ are, respectively, the voltage and current (at the position $\bar{x}$) transformed with respect to time. In (3) and throughout this paper, the subscript $\bar{x}$ accompanied with prime is used to indicate the partial differentiation with respect to $\bar{x}$, for example, $\bar{V}_{\bar{x}}'$ and $\bar{V}_{\bar{x}\bar{x}}''$ denote $d\bar{V}/d\bar{x}$ and $d^2\bar{V}/d\bar{x}^2$, respectively.

Combining the equations in (3), the result is

$$\{\sqrt{\bar{C}(\bar{x})}\,\bar{V}(\bar{x}, s)\}_{\bar{x}\bar{x}}''+\{\Lambda(s)-\Delta^C(\bar{x})\}\sqrt{\bar{C}(\bar{x})}\,\bar{V}(\bar{x}, s)=0 \tag{4a}$$

or

$$\{\sqrt{\bar{L}(\bar{x})}\,\bar{I}(\bar{x}, s)\}_{\bar{x}\bar{x}}''+\{\Lambda(s)-\Delta^L(\bar{x})\}\sqrt{\bar{L}(\bar{x})}\,\bar{I}(\bar{x}, s)=0 \tag{4b}$$

where

$$\Lambda(s)=-(s+r)(s+g)\{\int_0^l \sqrt{L(x)C(x)}dx\}^2 \tag{5}$$

$$\Delta^C(\bar{x})=\{\sqrt{\bar{C}(\bar{x})}\}_{\bar{x}\bar{x}}''/\sqrt{\bar{C}(\bar{x})} \tag{6a}$$

$$\Delta^L(\bar{x})=\{\sqrt{\bar{L}(\bar{x})}\}_{\bar{x}\bar{x}}''/\sqrt{\bar{L}(\bar{x})} \tag{6b}$$

The two-port parameters can be described by either the port voltages or the port currents. For instance, the short-circuit driving-point admittance function $y_{11}(s)$ can be expressed in terms of the port voltages,

$$y_{11}(s) = -\bar{V}_{\bar{x}}'(0, s) / (s+r)\,\bar{L}(0)\,\bar{V}(0, s)\,|_{\bar{V}(1,\, s)=0} \tag{7}$$

The pole-zero distribution of $y_{11}(s)$ can be established from the Sturm-Liouville equation (4a) with the boundary conditions,

$$\bar{V}(0, s) = \bar{V}(1, s) = 0 \quad \text{(for poles)} \tag{8}$$

and

$$\bar{V}_{\bar{x}}'(0, s) = \bar{V}(1, s) = 0 \quad \text{(for zeros)} \tag{9}$$

In both of the Sturm-Liouville problem, (4a), (8), and the problem, (4b), (9), the eigenvalues are positive real numbers except the trivial case of $\Lambda=0$, and are simple, isolated and infinite in number having no accumulation point along the positive real axis of the complex Λ-plane [2-3]. Let $\Lambda^p{}_n$ and $\bar{V}^p{}_n(\bar{x})$ be, respectively, the nth nontrivial eigenvalue and the corresponding eigenfunction of the problem, (4a), (8). Similarly, let $\Lambda^z{}_n$ and $\bar{V}^z{}_n(\bar{x})$ be, respectively, the nth eigenvalue and the corresponding eigenfunction of the problem, (4a), (9). The eigenvalues are separated as

$$\Lambda^z{}_n < \Lambda^p{}_n < \Lambda^z{}_{n+1} \tag{10}$$

for any positive integer n. The eigenvalues, $\Lambda^p{}_n$ and $\Lambda^z{}_n$, approach to, respectively, $(n\pi)^2$ and $\{(n-\frac{1}{2})\pi\}^2$ with sufficiently increasing n [3]. These properties also are true for any driving-point immittance.

By the aid of the Weierstrass factor theorem, the short-circuit driving-point admittance function $y_{11}(s)$ can be written as the infinite products,

$$y_{11}(s) = K\prod_{n=1}^{\infty}\{\Lambda(s)-\Lambda^z{}_n\} \Big/ (s+r)\prod_{n=1}^{\infty}\{\Lambda(s)-\Lambda^p{}_n\} \tag{11}$$

where K is a constant. The critical frequency s_n corresponding to nth eigenvalue Λ_n can be determined from (5).

Let
$$\Lambda^C = \{\tfrac{1}{2}(r-g)\int_0^l \sqrt{L(x)C(x)}\,dx\}^2 \tag{12}$$

Three cases are considered in the relation between Λ_n and Λ^C. First, when an eigenvalue Λ_n is equal to Λ^C, Λ_n has one to one correspondence to the critical frequency,

$$s_n = -\tfrac{1}{2}(r+g) \tag{13}$$

Here, the critical frequency is located on the negative real axis on the s-plane.

Secondly, when an eigenvalue Λ_n is less than Λ^C, Λ_n has one to one correspondence to a pair of critical frequencies,

$$s_n = -\tfrac{1}{2}(r+g) \pm \sqrt{\Lambda^C-\Lambda_n} \Big/ \int_0^l \sqrt{L(x)C(x)}\,dx \tag{14}$$

Here, the critical frequencies are located symmetrically along the negative real axis on both sides of $s=-\frac{1}{2}\,(r+g)$.

Thirdly, when an eigenvalue Λ_n is more than Λ^C, Λ_n has one to one correspondence to a pair of critical frequencies,

$$s_n = -\tfrac{1}{2}(r+g) \pm j\sqrt{\Lambda_n - \Lambda^C} \,/ \int_0^l \sqrt{L(x)C(x)}\,dx \tag{15}$$

Here, the critical frequencies are complex conjugates with the constant real part $-\tfrac{1}{2}(r+g)$.

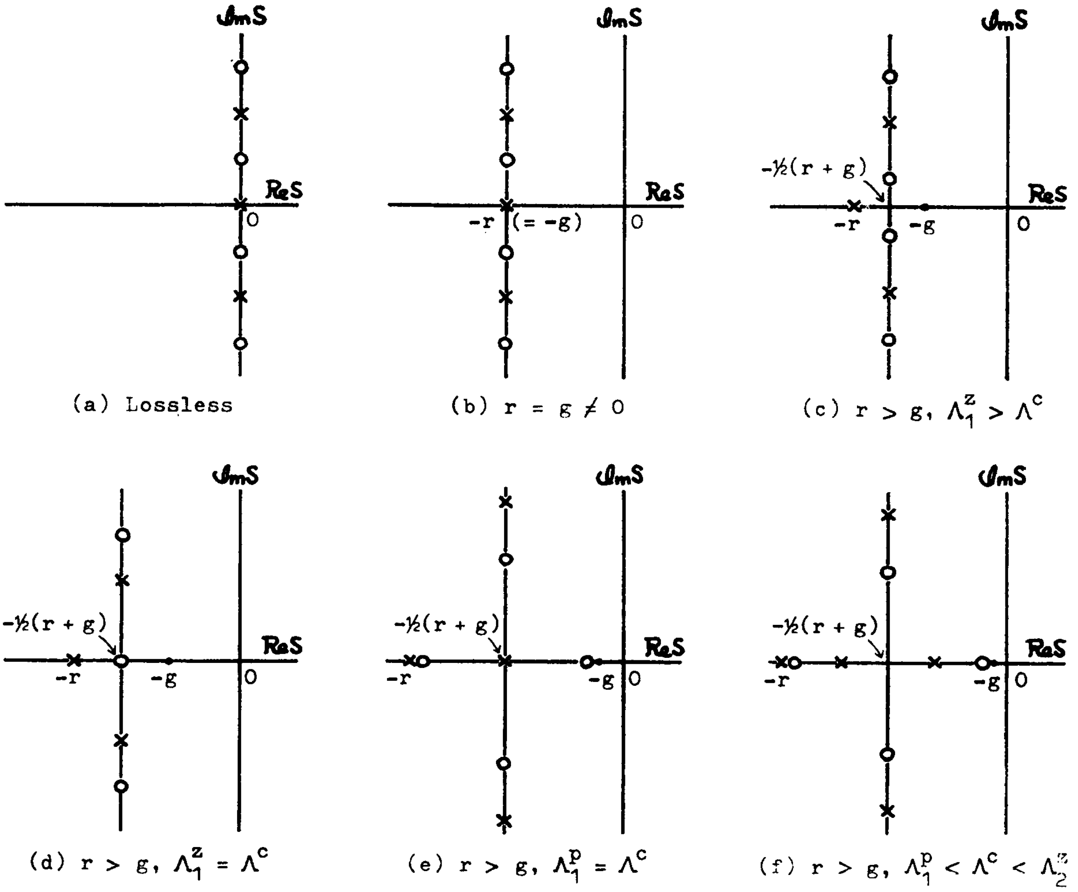

Fig. 1 Some typical pole-zero pistributions of the short-circuit driving-point admittance functions of pseudo-distortionless transmission lines of finite length. The symbols, × and ○, denote pole and zero, respectively.

The pole-zero distribution of $y_{11}(s)$ of a distortionless transmission line, for which $r=g$, is illustrated in Fig. 1a for a lossless case, and in Fig. 1b for a lossy case. In either case, the single pole is located at $s=-r$, while the conjugate poles and zeros (infinite in number) are distributed along the line of Re $s=-r$ $(=-g)$.

In a more general case of $r \neq g$, a single pole also is located at $s=-r$. Some pairs of poles and some pairs of zeros are distributed symmetrically along the negative real axis on both sides of $s=-\tfrac{1}{2}(r+g)$, while the remaining conjugate poles and zeros (infinite in number) are along the line of Re $s=-\tfrac{1}{2}(r+g)$. Here it is interesting that the frequency, $s=-g$, is neither a pole nor a zero [4]. When

$$|r-g| < 2\sqrt{\Delta^z{}_1} \Big/ \int_0^l \sqrt{L(x)C(x)}\,dx,$$

all the poles and zeros except the pole, $s=-r$, are distributed on the line of $\mathrm{Re}\, s = -\frac{1}{2}(r+g)$. If

$$|r-g| = 2\sqrt{\Delta^p{}_n} \Big/ \int_0^l \sqrt{L(x)C(x)}\,dx,$$

then $s=-\frac{1}{2}(r+g)$ is the location of a pair of overlapped nth poles. Similarly, if

$$|r-g| = 2\sqrt{\Delta^z{}_n} \Big/ \int_0^l \sqrt{L(x)C(x)}\,dx,$$

then $s=-\frac{1}{2}(r+g)$ is the location of a pair of overlapped nth zeros. Some typical pole-zero distributions in the case of $r \neq g$ are shown in Fig. 1c–f.

3. GENERALIZED EXPONENTIAL LINE SECTIONS

A transmission line in which either $\Delta^C(\bar{x})$ or $\Delta^L(\bar{x})$ is independent of $\bar{x}$ is called a generalized exponential line [1]. When

$$\Delta^C(\bar{x}) = \Delta^C = \text{constant} \tag{16}$$

and both of $\bar{C}(0)$ and $\bar{C}(1)$ are given, the distributed shunt capacitance function can be written as

$$\bar{C}(\bar{x}) = \mathrm{csch}^2\sqrt{\Delta^C}\,\{\sqrt{\bar{C}(1)}\,\sinh\sqrt{\Delta^C}\,\bar{x} + \sqrt{\bar{C}(0)}\,\sinh\sqrt{\Delta^C}\,(1-\bar{x})\}^2 \tag{17}$$

where Δ^C is available at values more than $-\pi^2$, or

$$\Delta^C > -\pi^2 \tag{18}$$

Similarly, in the case of

$$\Delta^L(\bar{x}) = \Delta^L = \text{constant} \tag{19}$$

and both of $\bar{L}(0)$ and $\bar{L}(1)$ are given, the distributed series inductance function is

$$\bar{L}(\bar{x}) = \mathrm{csch}^2\sqrt{\Delta^L}\,\{\sqrt{\bar{L}(1)}\,\sinh\sqrt{\Delta^L}\,\bar{x} + \sqrt{\bar{L}(0)}\,\sinh\sqrt{\Delta^L}\,(1-\bar{x})\}^2 \tag{20}$$

where Δ^L also is available at values over $-\pi^2$, or

$$\Delta^L > -\pi^2 \tag{21}$$

Defining the impedance level $\bar{Z}_0(\bar{x})$ at position $\bar{x}$ in the generalized exponential line to be

$$\bar{Z}_0(\bar{x}) = \sqrt{\bar{L}(\bar{x})/\bar{C}(\bar{x})}, \tag{22}$$

it is clear from (2) that

$$\bar{Z}_0(1)/\bar{Z}_0(0) = \bar{C}(0)/\bar{C}(1) = \bar{L}(1)/\bar{L}(0) \tag{23}$$

For notational convenience, let α denote the impedance ratio at the line ends, or

$$\alpha = \bar{Z}_0(1)/\bar{Z}_0(0) \tag{24}$$

In the generalized exponential lines, the line of

$$\Delta^C = \Delta^L = (\ln\sqrt{\alpha})^2$$

is an exponential line, and the line of

$$\Delta^C = 0 \quad \text{or} \quad \Delta^L = 0$$

is a quadratic line, and especially the exponential line or the quadratic line of

$$\alpha = 1$$

is a uniform line. In addition to these three lines, all of lines termed a squared-hyperbolic-cosine line, a squared-hyperbolic-sine line, and a squared-cosine line also come within the category of the generalized exponential lines. Namely, all of these six lines are the special cases of the generalized exponential lines.

As an example for examining the pole-zero distribution of the driving-point immittance function of a transimission line of finite lenth, we deal with the transmission line of which the distributed shunt capacitance function is given by (17). The short-circuit driving-point admittance function $y_{11}(s)$ of the two-port can be expressed by [1]

$$y_{11}(s) = [\sqrt{\Delta^C - \Lambda(s)}\coth\sqrt{\Delta^C - \Lambda(s)} + \{\ln\sqrt{\bar{C}(0)}\}_{\bar{x}}'] / (s+r)\bar{L}(0) \qquad (25)$$

A. The Poles of $y_{11}(s)$

It is obvious from (25) that any eigen value of the Sturm-Liouville problem, (4a), (8), is more than Δ^C. The eigenvalues and the corresponding eigenfunctions are, respectively,

$$\Lambda^p{}_n = \Delta^C + (n\pi)^2 \qquad (26)$$

and

$$\bar{V}^p{}_n(\bar{x}) = \sin\{n\pi(1-\bar{x})\}/\sqrt{\bar{C}(\bar{x})} \qquad (27)$$

where n denotes any positive integer.

B. The Zeros of $y_{11}(s)$

Under the separation condition of (10), two cases on eigenvalues are considered: One is the case of

$$\Delta^C < \Lambda^z{}_1$$

and the other is the case of

$$0 < \Lambda^z{}_1 \leq \Delta^C < \Lambda^z{}_2.$$

When any eigenvalue is more than Δ^C, the eigenvalue and the corresponding eigenfunction are, respectively,

$$\Lambda^z{}_n = \Delta^C + \Gamma_n{}^2 \qquad (28)$$

and

$$\bar{V}^z{}_n(\bar{x}) = \sin\{\Gamma_n(1-\bar{x})\}/\sqrt{\bar{C}(\bar{x})} \qquad (29)$$

where n is any positive integer, and Γ_n denotes the nth root of the transcendental equation of Γ,

$$\Gamma + \{\ln\sqrt{\bar{C}(0)}\}_{\bar{x}}' \tan\Gamma = 0$$

When only the minimum eigenvalue is less than Δ^C or equal to Δ^C, the eigenvalues and the corresponding eigenfunctions are, respectively,

$$\begin{aligned} \Lambda^z{}_1 &= \Delta^C - \Gamma_1{}^2 \\ \Lambda^z{}_n &= \Delta^C + \Gamma_n{}^2 \end{aligned} \tag{30}$$

and

$$\begin{aligned} \bar{V}^z{}_1(\bar{x}) &= \sinh\{\Gamma_1(1-\bar{x})\} \,/\sqrt{\bar{C}(\bar{x})} \\ \bar{V}^z{}_n(\bar{x}) &= \sin\{\Gamma_n(1-\bar{x})\} \,/\sqrt{\bar{C}(\bar{x})} \end{aligned} \tag{31}$$

where n is any positive integer except 1, and Γ_1 denotes the root of the transcendental equation of Γ,

$$\Gamma + \{\ln\sqrt{\bar{C}(0)}\}_{\bar{x}}' \tanh\Gamma = 0$$

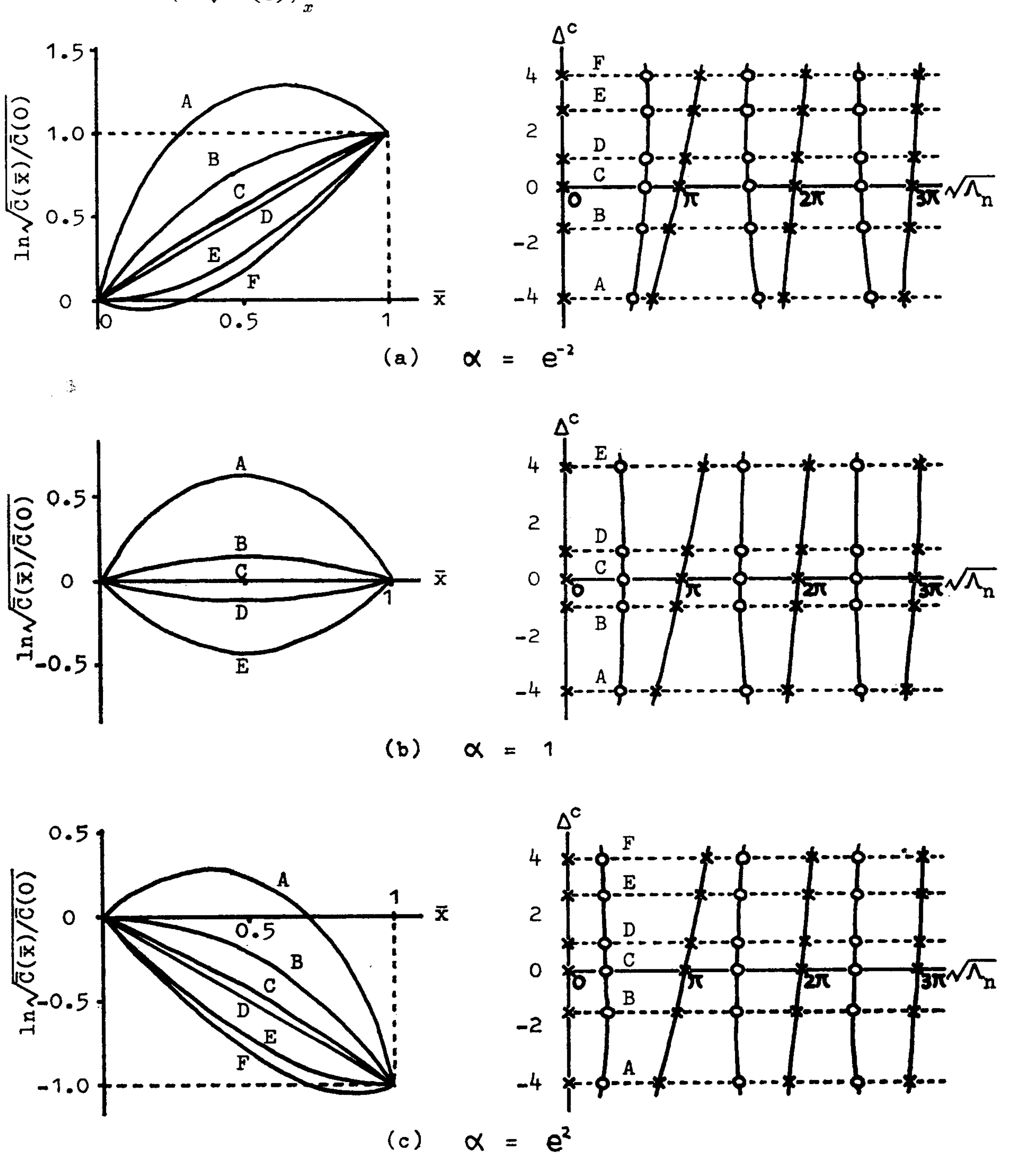

Fig. 2 The shapes of the distributed shunt capacitance function $\bar{C}(\bar{x})$ corresponding to various values for Δ^c, and the eigenvalues, $\Lambda^p{}_n$ and $\Lambda^z{}_n$, as functions of Δ^c.

and Γ_n denotes the n-1th root of the equation of Γ,

$$\Gamma + \{\ln\sqrt{\bar{C}(0)}\}_{\bar{x}}' \tan\Gamma = 0$$

In the three cases of $\alpha = e^{-2}$, 1, and e^2, the shapes of the distributed shunt capacitance function $\bar{C}(x)$ corresponding to various values for Δ^C, and the eigenvalues, $\Lambda^p{}_n$ and $\Lambda^z{}_n$, as functions of Δ^C are illustrated in Fig. 2.

As is obvious from Fig. 2, $\Lambda^z{}_n$ is influenced considerably by both of Δ^C and α, while $\Lambda^p{}_n$ is dependent on Δ^C but independent of α. The effect of Δ^C on each of $\Lambda^p{}_n$ and $\Lambda^z{}_n$ decreases with increasing n, and the effect of α on $\Lambda^z{}_n$ also decreases as n increases. Every $\Lambda^p{}_n$ is a monotonously increasing function of Δ^C, while any $\Lambda^z{}_n$ has an extremal value. $\Lambda^z{}_1$ has a local maximum, and any $\Lambda^z{}_n$ except $\Lambda^z{}_1$ has a local minimum. Any $\Lambda^z{}_n$ takes the extremal value, if and only if

$$\{\bar{Z}_0(0)\}_{\bar{x}}' = 0 \tag{32}$$

The condition in (32) is established when

$$\Delta^C = \begin{cases} -(\cos^{-1}\sqrt{\alpha^{-1}})^2 & \text{for } \alpha > 1 \\ 0 & \text{for } \alpha = 1 \\ (\cosh^{-1}\sqrt{\alpha^{-1}})^2 & \text{for } \alpha < 1 \end{cases} \tag{33}$$

Any $\Lambda^z{}_n$ shifts to smaller value with increasing α. When α is extremely large, $\Lambda^z{}_n$ takes a value nearly $\Lambda^p{}_{n-1}$. On the contrary, if α is extremely small, then $\Lambda^z{}_n$ takes a value nearly $\Lambda^p{}_n$. In either case, the separation condition in (10) is valid.

4. CONCLUSIONS

To simplify the model in question, this paper has used the electrically modified length instead of the physical length. Finding the pole-zero distribution of the driving-point immittance function for a transmission line of finite length results in solving the reduced Sturm-Liouville problems.

Except for a sigle pole at $s = -r$, the distribution of the critical frequencies of a driving-point admittance function is classiffied roughly into three cases, that is, some symmetrical pairs of poles and zeros along the negative real axis on both sides of $s = -\frac{1}{2}(r+g)$, conjugate poles and zeros (infinite in number) along the line of $\mathrm{Re}\, s = -\frac{1}{2}(r+g)$, and a double pole or a double zero at $s = -\frac{1}{2}(r+g)$. By examining whether the nth eigenvalue $\Lambda^p{}_n$ (or $\Lambda^z{}_n$) is more than a critical value Λ^C or less than Λ^C, it is possible to establish the case to which the practical distribution of a pair of the nth poles (or zeros) belongs.

In a generalized exponential line section, the constant Δ^C influences both of the eigenvalues, $\Lambda^p{}_n$ and $\Lambda^z{}_n$. Every $\Lambda^p{}_n$ shifts monotonously to large value with increasing Δ^C, while any $\Lambda^z{}_n$ has an extremal value. On the other hand, the impedance ratio α has an effect on only $\Lambda^z{}_n$. Every $\Lambda^z{}_n$ shifts to small value as α increases.

To sum up, the locus of the poles and zeros on the s-plane depends on both of r and g, and the critical frequencies on pole-zero distribution depend on the shape of the distributed series inductance and shunt capacitance.

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