Adjoint Transmission Line Sections

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Abstract

The two-port parameters of a tapered transmission line of finite length depends remarkably on the shape of the tapered impedance level. The sections with different-shaped tapers, generally, exhibit different transmission characteristics. Since there are all kinds of the shapes of the tapered impedance levels, the transmission characteristics also are multifarious granted that the impedance ratio and the line length are given. A pair of so-called adjoint transmission line sections in narrow sense, however, exhibits the same transmission characteristic in spite of the different-shaped tapers provided that $\kappa=0$. That is, the shapes of the tapered impedance levels have two to one correspondence to the transmission characteristic. This paper elucidates the relation between the shapes of the tapered impedance levels in the adjoint transmission lines.

1. Introduction

Many devices require a specific value of load impedance for optimum operation, but the impedance of the actual load that is to dissipate the power may differ widely from this value. So many varieties of impedance transformers have been used to change the actual load into an impedance of the desired value. In the majority of cases, the effective bandwidth of frequencies is very narrow.

In microwave circuits, the exponentially tapered transmission line of finite length has found wide application as a matching device suitable for matching two unequal impedances over a wide band of frequencies [1]. The exponential taper, however, is not always optimum in all respects. In this sense, when an arbitrarily tapered transmission line of finite length is used for impedance-matching purpose, it is interesting to find the relation between the matching characteristic and the shape of the tapered impedance level.

The shape of the tapered impedance level in a transmission line section influences strongly the two-port parameters of the section [3]. The sections with different-shaped tapers thus exhibit different transmission characteristcs. In a class of tapered transmission line sections, however, any section exhibits similar transmission characteristic in spite of the different-shaped tapers. The detailed discussion is found subsequently.

2. Adjoint Sections

Let N denote a section of tapered transmission line. When two ideal transformers and

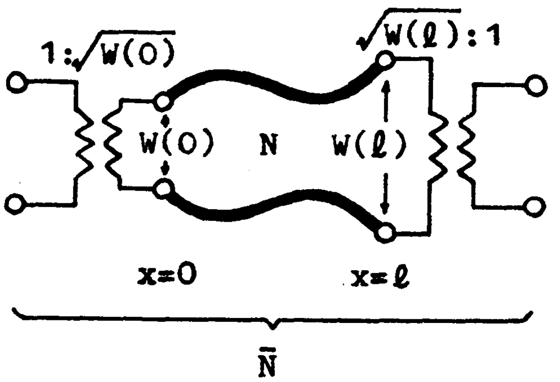

Fig. 1 The definition of $\bar{\mathrm{N}}$.

N are connected in cascade as shown in Fig. 1 so that each of the impedance levels at both ends of the overall section is normalized to 1 ohm, we designate the overall section by $\bar{\mathrm{N}}$.

If $\boldsymbol{F}$ is taken to denote the chain matrix of a section N of an arbitrarily tapered transmission line which exists for x defined on $0 \leqq x \leqq l$, then the chain matrix $\bar{\boldsymbol{F}}$ of $\bar{\mathrm{N}}$ can be written as

$$\bar{\boldsymbol{F}} = \boldsymbol{P}^{-1}(0)\boldsymbol{F}\boldsymbol{P}(l) \tag{1}$$

where

$$\boldsymbol{P}(x) = \begin{bmatrix} \sqrt{W(x)} & 0 \\ 0 & 1/\sqrt{W(x)} \end{bmatrix} \tag{2}$$

and $W(x)$ is the impedance level at position x in N.

We consider two transmission line sections, $\mathrm{N_A}$ and $\mathrm{N_B}$. $\mathrm{N_A}$ is defined as the adjoint of $\mathrm{N_B}$, provided that the chain matrix of $\bar{\mathrm{N}}_\mathrm{A}$ is equal to the transpose of the chain matrix of $\bar{\mathrm{N}}_\mathrm{B}$. Especially, if the chain matrix of $\bar{\mathrm{N}}_\mathrm{A}$ is symmetric, then one of the adioints of $\mathrm{N_A}$ is identical to $\mathrm{N_A}$ itself and $\mathrm{N_A}$ thus is termed to be self-adjoint.

When two adjoint transmission line sections are used for impedance-matching purpose, they will exhibit analogous matching characteristics for the similarity of their chain matrices. In this sense, it is intererting to clarify the relation between the shapes of the tapered impedance levels in the adjoint transmission line sections.

Let us now consider the transpose of the product of n symmetric matrices. If

$$\boldsymbol{A}_k = \boldsymbol{A}_k' \quad (k = 1, 2, \cdots, n)$$

then

$$(\boldsymbol{A}_1\boldsymbol{A}_2 \cdots \boldsymbol{A}_n)' = \boldsymbol{A}_n\boldsymbol{A}_{n-1} \cdots \boldsymbol{A}_1 \tag{3}$$

That is, the transpose of the product is the product of the transposes in reverse order. The same result as in (3) holds for infinite number of factors.

If a section N is taken to comprise the cascade of n self-adjoint sections, $\mathrm{N}_1, \mathrm{N}_2, \cdots, \mathrm{N}_n$, as shown in Fig. 2a, then the normalized section $\bar{\mathrm{N}}$ is equivalent to the cascade of n normalized sections, $\bar{\mathrm{N}}_1, \bar{\mathrm{N}}_2, \cdots, \bar{\mathrm{N}}_n$, as shown in Fig. 2b.

The adjoint of $\bar{\mathrm{N}}$ is equivalent to the cascade of $\bar{\mathrm{N}}'_k$s in reverse order as shown in Fig. 3a. That is, the kth section in Fig. 3a is $\bar{\mathrm{N}}_{n-k+1}$. This may be proved easily in terms of (3). Multiplying the internal impedance level of the k th section $(k=1, 2, \cdots, n)$ in Fig, 3a by $\eta \mathrm{W}_0\mathrm{W}_n/\mathrm{W}_{n-k}\mathrm{W}_{n-k+1}$, the turns ratios of the adjacent ideal transformers between the self-adjoint sections become reciprocal each other and the turns ratios of the endmost ideal transformers are reduced to

$$1 : \sqrt{\eta W_0} \quad \text{and} \quad \sqrt{\eta W_n} : 1,$$

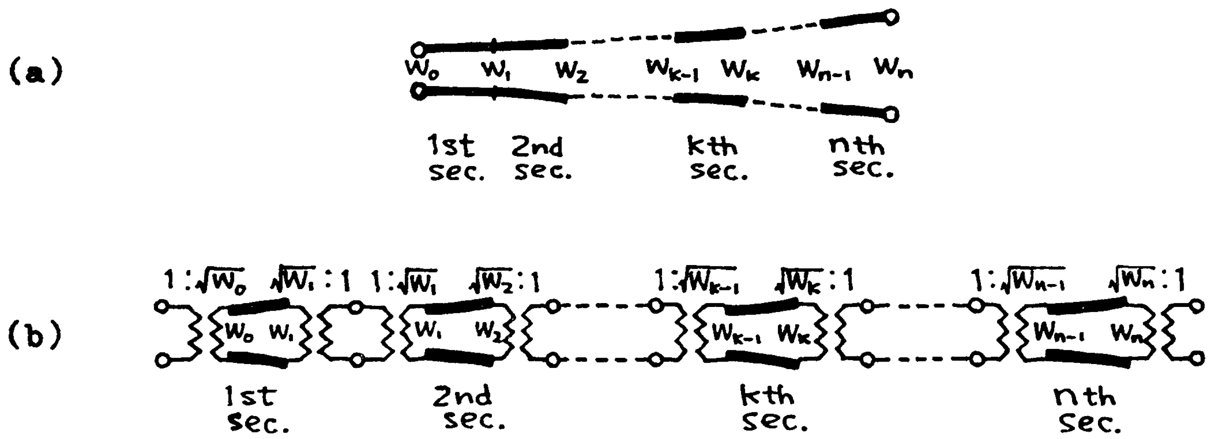

Fig. 2 (a) A section N composed of the cascade of n self-adjoint transmission line sections. (b) A section equivalent to the normalized section $\bar{N}$.

respectively, where η is any positive constant. All the intermediate ideal transformers thus can be omitted. Finally, we obtain the overall section in Fig. 3b as the adjoint of N. The impedance levels at both ends of the overall section in Fig. 3b are

$$\eta W_0 \quad \text{and} \quad \eta W_n ,$$

respectively, and the impedance levels at both ends of the k th section are

$$\eta W_0 W_n / W_{n-k+1} \quad \text{and} \quad \eta W_0 W_n / W_{n-k} ,$$

respectively.

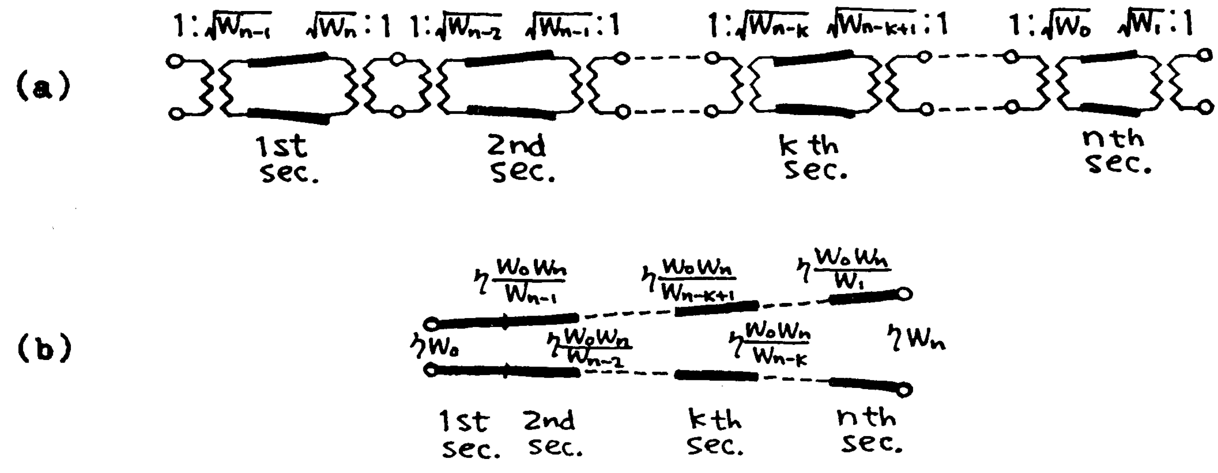

Fig. 3 (a) A section equivaleut to the adjoint of the overall section in Fig. 2b. (b) A section equivalent to the adjoint of the section N in Fig. 2a.

The solution to the problem on the cascade of n self-adjoint sections described above can be expanded to the general case that the impedance level of the overall section is given by a function of position x. Any section of a tapered transmission line may be approximated by the cascade of n self-adjoint sections, provided that the number of divisions n is large enough. Strictly speaking, a section with arbitrary taper may be defined as the limit when n approaches infinity. The adjoint of a transmission line section which exists for x defined on $0 \leqq x \leqq l$ thus can be described as the transmission line section whose impedance level is

$$\eta W(0) W(l) / W(l-x) \tag{4}$$

where $W(x)$ is the impedance level at position x in the original section and η is any positive constant.

If the equation

$$W(x) W(l-x) = W(0) W(l) \tag{5}$$

holds in a tapered transmission line section, then the section is a self-adjoint section. For example, any exponentially tapered section is one of the self-adjoint sections. In fact, the chain matrix of the normalized section of an exponential line is symmetric [2].

The expression in (4) signifies that there is an innumerable number of the sections adjoint to an original transmission line section. Among the numerous cases, the case of $\eta=1$ is especially important. In the case of $\eta=1$, two adjoint sections have the same impedance level at each of their corresponding ends, and they are called the adjoint sections in narrow sense.

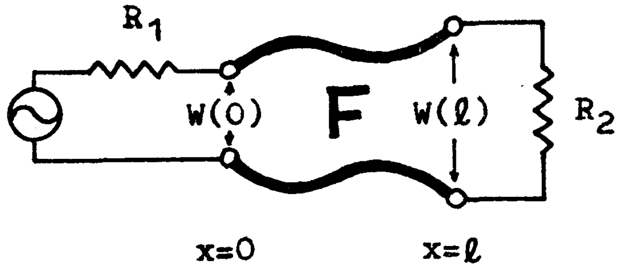

Fig. 4 An impedance-matching system, in which a tapered transmission line is used as a matching section.

The absolute value of the reflection coefficient at the input end of the impedance-matching section shown in Fig. 4 is given by [3]

$$|\Gamma| = \sqrt{\frac{\{(a-d)\cosh\tau + (a+d)\sinh\tau\}^2 + \{(b-c)\cosh\kappa + (b+c)\sinh\kappa\}^2}{\{(a+d)\cosh\tau + (a-d)\sinh\tau\}^2 + \{(b+c)\cosh\kappa + (b-c)\sinh\kappa\}^2}} \tag{6}$$

where

$$\tau = \frac{1}{2}\ln\{W(0)R_2/W(l)R_1\} \tag{7}$$

$$\kappa = \frac{1}{2}\ln\{W(0)W(l)/R_1R_2\} \tag{8}$$

and a, b, c and d are real numbers defined by

$$\begin{bmatrix} a & jb \\ jc & d \end{bmatrix} = \overline{\boldsymbol{F}} \tag{9}$$

When the impedance-matching section is replaced by the new section adjoint to the original, the matching characteristic of the new system is analogous to it of the original but for the difference of κ. Particularly, if the new section is the adjoint section in narrow sense and $\kappa=0$, then the matching characteristic is kept unchanged.

3. Conclusions

This paper has elucidated the relation between the shapes of the tapered impedance levels in the adjoint transmission line sections, and the evidence has been given that the adjoint of a transmission line section may be defined by the expression in (4). When each of two transmission line sections adjoint in narrow sense is used as an impedance-matching section, both of them exhibit the same matching characteristic in spite of the different-shaped tapers provided that $\kappa = 0$. Namely, the shapes of the tapered impedance levels have two to one correspondence to the matching characteristic, exclusive of the self-adjoint case in which the shape of the taper has one to one correspondence to the matching characteristic.

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