Generalized Greuling Goertzel Kernel and Generalized Wigner Kernel for the CM Angular Distribution and the Anisotropic Scattering

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Abstract

In order to take into account the effect of the scattering angular distribution in the center of mass system in calculating the angular neutron flux in fast reactor assembly we develop the synthetic kernel approximation. In this paper the generalized Greuling Goertzel kernel (G_n^{lm}) and the generalized Wigner kernel (W_n^{lm}) are proposed in order to estimate such effects. These generalized synthetic kernels enable us to allow rapid computations of collision integrals and are useful for calculations of the angular neutron fluxes in the vicinity of scattering resonances.

I. INTRODUCTION

The problems of determining fast neutron spectra in the fast reactor assembly have been studied for these three years and various approximations are in use to account for the scattering resonances and inelastic scattering. By expanding the total collision density, STACEY[1] applied continuous slowing down (CSD) theory to the treatment of the elastic moderation of fast neutrons in fast reactor assembly, where strong scattering resonances are prevalent. According to the approximate method proposed successfully by CADILHAC and PUJOL[2], DUNN and BECKER[3] calculated the fast neutron spectra by making use of a separable kernel for elastic scattering and inelastic scattering and VOLMERANGE[4] proposed a general model for the elastic and inelastic slowing down of fast neutrons.

There exists another approach to such problems, which is based on the synthetic kernel approximation (SKA). This approach is mainly developed by YAMAMURA and SEKIYA[5].

Almost all analyses mentioned above are performed under the assumption that the elastic angular distribution is isotropic in the center of mass (CM) system. Recntly the effect of the CM angular distribution on fast neutron moderation has been investigated by STACEY[6] according to his CSD theory. His approach is based on AMSTER[7] and GREULING and GOERTZEL [8]'s formalism who extended the original Greuling-Goertzel (GG) formalism[9] to accomodate anisotropic scattering and to determine the sensitivity of the approximation to various-order Legendre components of the angular scatterring data.

YAMAMURA and SEKIYA[10] have pointed out that it is difficult to take into account the effect of the CM scattering angular distribution upon the higher-order Legendre components of the angular neutron flux according to the GG approximation.

In order to avoid such difficulties, in this paper, we will propose the generalized synthetic kernels G_n^{lm} and W_n^{lm} for the CM angular distribution and the anisotropic scattering in the

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laboratory(L) system. Namely, the synthetic kernels G_n^{lm} ane W_u^{lm} will be defined for each coupling of the l'th-order Legendre component of the anisotropic scattering (L) and the m'th-order Legendre component of the CM angular distribution.

II. LETHARGY MOMENTS OF THE CM-TO-L TRANSFER FUNCTION

When calculating the angular spectrum in fast reactor assembly the anisotropy in the CM system must be taken into account. In attempting to study such problems most difficulties lie in the treatment of the collision integral. The l'th-order Legendre component of the elastic collision integral is defined as follows :

$$J_l(u)=\sum_i\int_{-\infty}^{u}du' h_i(u') f_{li}(u'\to u)\varphi_l(u'), \tag{1}$$

where $h_i(u)$ is the relative probability of elastic scattering for isotope i, $\varphi_l(u)$ is the l'th-order Legendre component of the angular collision density and $f_{li}(u'\to u)$ is the l'th-order Legendre component of the scattering transition probability of isotope i in the L system

$$f_{li}(u'\to u)=\frac{e^{-(u-u')}}{1-\alpha_i}P_l(\mu_L(u,u'))$$
$$\times\sum_{m=0}^{\infty}(2m+1)f_{mi}(u')P_m(\mu_c(u,u')) \quad u-\varepsilon_i<u'<u, \tag{2}$$

with $\alpha_i=(M_i-1)^2/(M_i+1)^2$, $\varepsilon_i=-\ln\alpha_i$ and M_i being the atomic mass of isotope i.
The coefficients $f_{mi}(u)$ are the m'th-order Legendre component of the scattering angular distribution in the CM system, while $\mu_L(u,u')$ and $\mu_c(u,u')$ are cosines of the scattering angle in the L and CM systems, respectively.

For simplicity we will omit the subscript i for isotope in the ensuing discussions.
When one attempts to calculate the angular neutron flux, it is a very important problem how to approximate the collision integral. The previous approximate slowing down theories[7),11)] were obtained by assuming $h(u')$ $\varphi_l(u')$ in the integrand of Eq. (1) to be slowly varying. In this paper we will assume $h(u')$ $\varphi_l(u')$ $f_m(u')$ to be a slowly varying function over the scattering interval

$$h(u')f_m(u')\varphi_l(u')=\sum_{k=0}^{\infty}\frac{(u'-u)^k}{k!}\mathbf{D}^k\,[h(u)f_m(u)\varphi_l(u)], \tag{3}$$

where $\mathbf{D}=\frac{d}{du}$.
The direct insertion of Eq. (3) into Eq. (1) yields

$$J_l(u)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}T_{lm}^k\,\mathbf{D}^k\,[h(u)f_m(u)\,\varphi_l(u)], \tag{4}$$

where T_{lm}^k is the well-known lethargy moment of the CM-to-L transfer function and it is defined as follows :

$$T_{lm}^k=\frac{(-1)^k(2m+1)}{k!\cdot(1-\alpha)}\int_0^{\varepsilon}u^k P_l(\mu_L(u))P_m(\mu_c(u))e^{-u}du. \tag{5}$$

In estimating the effects of the CM angular distribution upon the l'th-order Legendre component of the angular collision density, it is neccessary to know the appoximate values of T_{lm}^k, because the values of the T_{lm}^k have the one-to-one correspondence to such effects. By rough estimation we get the following mass-dependence of the quantities T_{lm}^k for a heavy-mass element :

$$T_{lm}^k = \begin{cases} e_{lm}^k \left(\frac{1}{M}\right)^k & |l-m| \leqq k \\ e_{lm}^k \left(\frac{1}{M}\right)^{|l-m|} & \text{otherwise,} \end{cases} \tag{6}$$

where the coefficient e_{lm}^k are smaller than unity, while e_{ll}^o is nearly equal to unity[11].

Equation (5) tells us that, even if within the GG approximation, the l'th-order Legendre component of the anisotropic scattering (L) may be influenced by the same order Legendre component of the CM angular distribution, if we does not account for the relative order of magnitude of the $f_m(u)$.

III. APPROXIMATE APPROACH TO DIFFERNTIAL POLYNOMIAL

Let us define the following differential polynomial which is appeared in Eq. (4) :

$$J_{lm}(\mathbf{D}) = \sum_{k=0}^{\infty} T_{lm}^k \mathbf{D}^k. \tag{6}$$

As was already pointed out by YAMAMURA and SEKIYA[12], the approximate procedure to obtain the generalized synthetic kernel for elastic scattering is equivalent to replacing this differential polynomial by the appropriate integral operator.

According to this theory, YAMAMURA and SEKIYA[13] have proposed the G_n kernel (the generalized GG kernel) for neutron slowing down in the case of the isotropic scattering in the CM system. Mathematically detailed examinations of YAMAMURA and SEKIYA's approach have revealed that there exists another type of the generalizd synthetic kernel, which will be considered as the generalization of the Wigner kernel.

In this paper we will extend such formalism for the isotropic scattering to accomodate the CM scattering angular distribution and the anisotropic scattering in the L system. The differential polynomial $J_{lm}(\mathbf{D})$ implies the contribution of the CM angular distribution to the l'th-order Legendre component of the anisotropic scattering in the L system. As was mentioned in chapter II, the higher order Legendre component of the CM angular distribution cannot be neglected in calculating the higher order Legendre component of the angular neutron flux.

Here we assume that the differential polynomial $J_{lm}(\mathbf{D})$ can be approximated by the following infinite continued fraction :

$$J_{lm}(\mathbf{D}) = \cfrac{T_{lm}^0}{1 + \cfrac{\alpha_1^{lm}\,\mathbf{D}}{1 + \cfrac{\alpha_2^{lm}\,\mathbf{D}}{1 + \cfrac{\alpha_3^{lm}\,\mathbf{D}}{1 + \cdots\cdots}}}} \quad \cdots\cdots, \tag{7}$$

where the quantities α_n^{lm} are the parameters which are expressed in terms of T_{lm}^k.

If we trancate the higher terms of this infinite continued fraction than α_{n+1}^{lm}, then we will obtain the following approximate expression for $J_{lm}(\mathbf{D})$:

$$J_{lm}(\mathbf{D}) \cong T_{lm}^0 \cdot \frac{Q_n^{lm}(\mathbf{D})}{P_n^{lm}(\mathbf{D})}. \tag{8}$$

From the property of the continued fraction, we can easily know that these differential polynomials $Q_n^{lm}(\mathbf{D})$ and $P_n^{lm}(\mathbf{D})$ satisfy the following relation :

$$P_n^{lm}(\mathbf{D}) = P_{n-1}^{lm}(\mathbf{D}) + \alpha_n^{lm}\,\mathbf{D}\cdot P_{n-2}^{lm}(\mathbf{D}), \tag{9}$$

$$Q_n^{lm}(\mathbf{D}) = Q_{n-1}^{lm}(\mathbf{D}) + \alpha_n^{lm}\,\mathbf{D}\cdot Q_{n-2}^{lm}(\mathbf{D}), \tag{10}$$

where the polynomials with lower indexes have the form

$$\begin{aligned}
&P_0^{lm}(\mathbf{D}) = 1, && Q_0^{lm}(\mathbf{D}) = 1,\\
&P_1^{lm}(\mathbf{D}) = 1 + \alpha_1^{lm}\,\mathbf{D}, && Q_1^{lm}(\mathbf{D}) = 1,\\
&P_2^{lm}(\mathbf{D}) = 1 + (\alpha_1^{lm} + \alpha_2^{lm})\,\mathbf{D}, && Q_2^{lm}(\mathbf{D}) = 1 + \alpha_2^{lm}\,\mathbf{D},\\
&P_3^{lm}(\mathbf{D}) = 1 + (\alpha_1^{lm} + \alpha_2^{lm} + \alpha_3^{lm})\,\mathbf{D} + \alpha_2^{lm}\,\alpha_4^{lm}\,\mathbf{D}^2, && Q_3^{lm}(\mathbf{D}) = 1 + (\alpha_2^{lm} + \alpha_3^{lm})\,\mathbf{D}.
\end{aligned} \tag{11}$$

Namely $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ are the polynomials of degrees $\left[\frac{n+1}{2}\right]$ and $\left[\frac{n}{2}\right]$, respectively.

Here we try to express explicitly the parameters α_k^{lm} in terms of the lethargy moments T_{lm}^k. For this purpose we will write down the polynomials $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ in the form

$$P_n^{lm}(\mathbf{D}) = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} p_{n,k}^{lm}\,\mathbf{D}^k, \tag{12}$$

$$Q_n^{lm}(\mathbf{D}) = \sum_{k=0}^{\left[\frac{n}{2}\right]} q_{n,k}^{lm}\,\mathbf{D}^k \tag{13}$$

where $p_{n0}^{lm} = q_{n0}^{lm} = 1$.

As is shown in Eq. (11), if n is even, the polynomials $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ are the polynomials of the same degree $n/2$, while $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ with odd n are the polynomials of degree $\frac{n+1}{2}$ and $\frac{n-1}{2}$, respectively. Then let us discuss the problem of obtaining the expression for α_n^{lm} by separating into two cases.

A) n=2J

In this case Eqs. (12) and (13) are rewritten by

$$P_{2J}^{lm}(\mathbf{D}) = \sum_{\lambda=0}^{J} p_{2J,\lambda}^{lm}\,\mathbf{D}^\lambda, \tag{12a}$$

$$Q_{2J}^{lm}(\mathbf{D}) = \sum_{\lambda=0}^{J} q_{2J,\lambda}^{lm}\,\mathbf{D}^\lambda. \tag{13a}$$

Substituting Eqs. (12a), (13a) and (6) into Eq. (8), we get

$$\sum_{k=0}^{\infty} T_{lm}^{k}\, \mathbf{D}^{k} \simeq T_{lm}^{0} \cdot \frac{\sum_{k=0}^{J} q_{2J,k}^{lm}\, \mathbf{D}^{k}}{\sum_{k=0}^{J} p_{2J,k}^{lm}\, \mathbf{D}^{k}}. \tag{14}$$

Operating $P_{2J}^{lm}(\mathbf{D})$ on both sides of Eq. (14) and comparing the coefficients of the same power of $\mathbf{D}$, we obtain the following simultaneous algebraic equations :

$$\sum_{\lambda=0}^{\mu} p_{2J,\lambda}^{lm}\, T_{lm}^{\mu-\lambda} = T_{lm}^{0}\, q_{2J,\mu}^{lm} \qquad \mu=0, 1, \cdots J, \tag{15}$$

$$\sum_{\lambda=0}^{J} p_{2J,\lambda}^{lm}\, T_{lm}^{\mu-\lambda} = 0 \qquad \mu=J+1,\ J+2, \cdots, 2J. \tag{16}$$

The unknown coefficients $p_{2J,\lambda}^{lm}$ are easily determined from the set of Eq. (16). Namely these coefficients can be written by means of determinants

$$p_{2J,\lambda}^{lm} = \frac{1}{D_{J,1}^{lm}}\, D_{J+1,1}^{lm}(\underbrace{0, 0, \cdots 0,}_{\lambda+1} 1, 0 \cdots 0), \tag{17}$$

where

$$\mathrm{D}_{J,K}^{lm} = \begin{vmatrix} T_{lm}^{J+K-1}, & T_{lm}^{J+K-2}, & \cdots & T_{lm}^{K+1}, & T_{lm}^{K} \\ T_{lm}^{J+K} & & & T_{lm}^{K+2}, & T_{lm}^{K+1} \\ \vdots & & & \vdots & \vdots \\ T_{lm}^{2J+K-2}, & & & T_{lm}^{J+K} & T_{lm}^{J+K-1} \end{vmatrix} \tag{18}$$

$$D_{J,K}^{lm}(a_1, a_2, \cdots, a_J) = \begin{vmatrix} a_1 & a_2 & & a_{J-1} & a_J \\ T_{lm}^{J+K-1}, & T_{lm}^{J+K-2}, & \cdots & T_{lm}^{K+1} & T_{lm}^{K} \\ T_{lm}^{J+K}, & T_{lm}^{J+K-1}, & \cdots & & T_{lm}^{K+1} \\ \vdots & \vdots & & & \vdots \\ T_{lm}^{2J+K-3}, & T_{lm}^{2J+k-4}, & \cdots & & T_{lm}^{J+K-2} \end{vmatrix} \tag{19}$$

and $D_{O,K}^{lm}=1$.

The substitution of Eq. (17) into Eq. (15) yields the following expression for the coefficients $q_{2J,\mu}^{lm}$:

$$q_{2J,\mu}^{lm} = \frac{1}{T_{lm}^{o}\, D_{J,1}^{lm}}\, D_{J+1,1}^{lm}(T_{lm}^{\mu},\ T_{lm}^{\mu-1}, \cdots T_{lm}^{o}, 00\cdots 0). \tag{20}$$

As is known from Eq. (12), one can write down the explicit expression for the polynomial $P_{2J}^{lm}(\mathbf{D})$ by means of the determinant, namely

$$P_{2J}^{lm}(\mathbf{D}) = \frac{1}{D_{J,1}^{lm}}\, D_{J+1,1}^{lm}(1, \mathbf{D}, \mathbf{D}^{2}, \cdots \mathbf{D}^{J}). \tag{21}$$

B) $n=2J\text{-}1$

In this case Equations (12) and (13) become

$$P_{2J-1}^{lm}(\mathbf{D}) = \sum_{\lambda=0}^{J} p_{2J-1,\lambda}^{lm}\, \mathbf{D}^{\lambda}, \tag{12b}$$

$$Q_{2J-1}^{lm}(\mathbf{D}) = \sum_{\lambda=0}^{J-1} q_{2J-1,\lambda}^{lm}\, \mathbf{D}^{\lambda}. \tag{13b}$$

The similar procedure to the case of $n=2J$ yields the following simultaneous equation for the unknown coefficients :

$$\sum_{\lambda=0}^{\mu} p_{2J-1,\lambda}^{lm}\ T_{lm}^{\mu-\lambda} = T_{lm}^{0}\ q_{2J-1,\lambda}^{lm} \qquad \mu=0, 1, \cdots J-1, \tag{22}$$

$$\sum_{\lambda=0}^{J} p_{2J-1,\lambda}^{lm}\ T_{lm}^{\mu-\lambda} = 0 \qquad \mu=J, J+1, \cdots, 2J-1, \tag{23}$$

By solving this simultaneous equation we know that the coefficients $p_{2J-1,\lambda}^{lm}$ and $q_{2J-1,\lambda}^{lm}$ have the explicit expressions of the form

$$p_{2J-1,\lambda}^{lm} = \frac{1}{D_{J,0}^{lm}}\ D_{J+1,0}^{lm}\ (\underbrace{0, 0, \cdots 0, 1}_{\lambda+1}, 0 \cdots 0), \tag{24}$$

$$q_{2J-1,\lambda}^{lm} = \frac{1}{T_{lm}^{0} \cdot D_{J,0}^{lm}}\ D_{J+1,0}^{lm} (T_{lm}^{\lambda},\ T_{lm}^{\lambda-1}, \cdots T_{lm}^{0}, 0, \cdots 0), \tag{25}$$

while the polynomial $P_{2J-1}^{lm}(\mathbf{D})$ is represented as

$$P_{2J-1}^{lm}(\mathbf{D}) = \frac{1}{D_{J,0}^{lm}}\ D_{J+1,0}^{lm} (1, \mathbf{D}, \mathbf{D}^2, \cdots \mathbf{D}^J). \tag{26}$$

Here we have got the explicit expression for coefficints of $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$. By substituting Eq. (12) with the coefficients defined in Eqs. (17) and (24) into Eq. (9) and comparing the coefficient of the same power of $\mathbf{D}$ in both sides of the rearranged equation, we can easily obtain the expressins for α_n^{lm}, i. e.,

$$\alpha_{2J}^{lm} = \frac{D_{J-1,1}^{lm}}{D_{J-1,2}^{lm}} \left\{ \frac{D_{J,2}^{lm}}{D_{J,1}^{lm}} - \frac{D_{J,1}^{lm}}{D_{J,0}^{lm}} \right\} \qquad J \geqq 1, \tag{27}$$

$$\alpha_{2J-1}^{lm} = \frac{D_{J-1,0}^{lm}}{D_{J-1,1}^{lm}} \cdot \frac{D_{J,1}^{lm}}{D_{J,0}^{lm}} \qquad J \geqq 1. \tag{28}$$

In terms of the parameter α_n^{lm} thus obtained the polynomials $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ can explicitly be expressed according to Eqs. (9) and (10), which seems to be useful for the analytical treatment of slowing down problem like calculation of the Placzek function.

It is mentioned before that the procedure of replacing the differential polynomial $J_{lm}(\mathbf{D})$ by the integral operal operator such as Eq. (8) corresponds to approximating the exact scattering kernel Eq. (2) by the corresponding synthetic kernel. Namely if $J_{lm}(\mathbf{D})$ are approximated by $T_{lm}^{0} \cdot Q_n^{lm}(\mathbf{D})/P_n^{lm}(\mathbf{D})$, the first n lethargy moments of thus obtained synthetic kernel are consistent with those of the exact scattering kernel(13). Therefore the degree n of these polynomials implies the order of correctness of the synthetic kernel. Consequently speaking, the approximation by the polynomials $P_n^{lm}(\mathbf{D})$ and $Q_n^{lm}(\mathbf{D})$ with even degree corresponds to the generalized Greuling Goertzel kernel and that by the polynomials with odd degree to the generalized Wigner kernel.

IV. DERIVATION OF GENERALIZED SYNTHETIC KERNELS

The synthetic kernel approximation (SKA) method for neutron slowing down is originated at the work of GREULING et al[9]. Great efforts have been employed in extending the original theory to accomodate the effect of the various order Legendre components of the CM angular distribution upon the anisotropic scattering in the L system[7][11].

As was mentioned in chapter II, the previous approaches for the SKA are characterized by the Taylor expansion of the integrand $h(u')\psi(u')$ in Eq. (1) about $u'=u$, while the present procedure starts with that of $h(u')f_m(u')\psi(u')$ about $u'=u$. As a result, the slowing down parameters $\lambda_L(u)$ (Ferziger and Zweifel's notation[11]) defined by the former approaches are functions of lethargy, which give rise to some difficulties in estimating the effect of the CM angular distribution on the higher order Legendre component of the anisotropic scattering in the L system.

According to the present approximation, the different kernel is defined for the different coupling of the Legendre components of the CM angular distribution and the anisotropic scattering in the L system. Therefore such difficulties do not appear in the present approach.

From Eqs. (1) and (4), the scattering kernel $f_l(u'\rightarrow u)$ can be defined in another form

$$f_l(u'\rightarrow u)=\sum_{m=0}^{\infty} J_{lm}(\mathbf{D})\cdot f_m(u)\delta(u-u'). \tag{29}$$

As was mentioned in chapter III, the approximate procedure for $f_l(u'\rightarrow u)$ is associated with replacing the differential polynomial $J_{lm}(\mathbf{D})$ by means of the integral operator $T^0_{lm}\,Q^{lm}_n(\mathbf{D})/P^{lm}_n(\mathbf{D})$. In chapter III, we discussed mainly the properties of these differential polynomials. In this chapter we will derive the synthetic kernel for such approximate procedure. It was also pointed out that the properties of the polynomials $P^{lm}_n(\mathbf{D})$ and $Q^{lm}_n(\mathbf{D})$ are characterized according to whether the degree n of these polynomials is odd or even. Namely, this fact implies that the synthetic kernel approximated by the polynomials with even degree is of different type from that approximated by those with odd degree. Therefore we will discuss the procedure of deriving the synthetic kernel by separating into two cases.

A) $n=2J$

In this case the polynomials $P^{lm}_{2J}(\mathbf{D})$ and $Q^{lm}_{2J}(\mathbf{D})$ are of the same degree. Therefore Equation (14) are rearranged as follows :

$$\sum_{k=0}^{\infty} T^k_{lm}\,\mathbf{D}^k \simeq r^{lm}_J+v^{lm}_{2J}\cdot R^{lm}_{2J}(\mathbf{D}), \tag{30}$$

where

$$R^{lm}_{2J}(\mathbf{D})=\frac{\sum_{\lambda=0}^{J-1}\sigma^{lm}_{2J,\lambda}\,\mathbf{D}^\lambda}{\sum_{\lambda=0}^{J}\nu^{lm}_{2J,\lambda}\,\mathbf{D}^\lambda}, \tag{31}$$

and $\sigma^{lm}_{2J,J-1}=\nu^{lm}_{2J,J}=1$. The parameters r^{lm}_J, v^{lm}_{2J}, $\sigma^{lm}_{2J,\lambda}$ and $\nu^{lm}_{2J,\lambda}$ can be expressed in terms of the proper determinants like $p^{lm}_{n,\lambda}$ and $q^{lm}_{n,\lambda}$. Of cource these parameters can be represented in terms of α_n^{lm}. For the practical calculation of these parameters, the determinant representation is more useful. The simultaneous equations for these parameters are written by

$$\sum_{\mu=0}^{\lambda} \nu_{2J,\mu}^{lm} T_{lm}^{\lambda-\mu} = v_{2J}^{lm} \cdot \sigma_{2J,\lambda}^{lm} + r_J^{lm} \cdot \nu_{2J,\lambda}^{lm} \qquad \lambda = 0, 0, \cdots J, \tag{32}$$

$$\sum_{\mu=0}^{J} \nu_{2J,\mu}^{lm} \cdot T_{lm}^{\lambda-\mu} = 0 \qquad \lambda = J+1, J+2, \cdots 2J \tag{33}$$

The coefficients $\nu_{2J,\lambda}^{lm}$ are determined from Eq. (33), i.e.,

$$\nu_{2J,\lambda}^{lm} = \frac{(-1)^J}{D_{J,2}^{lm}} D_{J+1,1}^{lm}(0, 0, \cdots 0, 1, \cdots 0) \tag{34}$$

The unknowns r_{2J}^{lm} and v_{2J}^{lm} are obtained by setting $\lambda=J$ and $\lambda=J-1$, respectively :

$$r_J^{lm} = (-1)^J \frac{D_{J+1,0}^{lm}}{D_{J,2}^{lm}}, \tag{35}$$

$$v_{2J}^{lm} = \frac{(-1)^{J+1}}{D_{J,2}^{lm}} D_{J+1,1}^{lm}(T_{lm}^J - T_{lm}^{J-1}, T_{lm}^{J-1} - T_{lm}^{J-2}, \cdots T_{lm}^1 - T_{lm}^0, T_{lm}^0), \tag{36}$$

while the coefficients $\sigma_{2J,\lambda}^{lm}$ are the following :

$$\sigma_{2J,\lambda}^{lm} = \frac{D_{J+1,1}^{lm}(T_{lm}^J - T_{lm}^{\lambda}, T_{lm}^{J-1} - T_{lm}^{\lambda-1}, \cdots T_{lm}^{J-\lambda} - T_{lm}^0, T_{lm}^{J-\lambda-1}, \cdots, T_{lm}^0)}{D_{J+1,1}^{lm}(T_{lm}^J - T_{lm}^{J-1}, T_{lm}^{J-1} - T_{lm}^{J-2}, \cdots T_{lm}^1 - T_{lm}^0, T_{lm}^0)} \tag{37}$$

Since the explicit expressions for the parameters in Eqs. (30) and (31), are given we can define the synthetic kernel for contribution of the m'th-order Legendre component of the angular distribution in the CM system to the l' th-order Legendre component of the anisotropic scattering in the L system :

$$G_J^{lm}(u' \to u) = r_J^{lm} \delta(u-u') + v_{2J}^{lm} \cdot R_{2J}^{lm}(\mathbf{D}) \cdot \delta(u-u'). \tag{38}$$

That is, this is the G_J^{lm} kernel which is the generalization of the Greuling Goertzel kernel.

In order to rewrite this G_J^{lm} kernel in terms of the well-known functions we assume that the second term on the right hand side of Eq. (38) has a unique composition into the sum of simple fractions, i. e',

$$R_{2J}^{lm}(\mathbf{D}) = \frac{l_{2J}^{lm}}{\mathbf{D}+\lambda_J^{lm}} + \sum_{\mu=1}^{[\frac{J}{2}]} (-1)^{\mu} \left\{ \frac{m_{2J,\mu}^{lm} + i\, n_{2J,\mu}^{lm}}{\mathbf{D}+\beta_{J,\mu}^{lm} + i\, \gamma_{F,\mu}^{lm}} + \frac{m_{2J,\mu}^{lm} - i\, n_{2J,\mu}^{lm}}{\mathbf{D}+\beta_{J,\mu}^{lm} - i\, \gamma_{J,\mu}^{\kappa m}} \right\}, \tag{39}$$

where

$$l_{2J}^{lm} = \{ (x+\lambda_J^{lm}) R_{2J}^{lm}(x) \}_{x=-\gamma_J^{lm}}, \tag{40}$$

$$m_{2J,\mu}^{lm} = (-1)^{\mu} Re\{x+\beta_{J,\mu}^{lm}+i\gamma_{J,\mu}^{lm}) R_{2J}^{lm}(x)\}_{x=-\beta_{J,\mu}^{lm}-i\gamma_{J,\mu}^{lm}}, \tag{41}$$

$$n_{2J,\mu}^{lm} = (-1)_{\mu}\, Im\{x+\beta_{J,\mu}^{lm}+i\gamma_{J,\mu}^{lm}) \cdot R_{2J}^{lm}(x)\}_{x=-\beta_{J,\mu}^{lm}-i\gamma_{J,\mu}^{lm}}. \tag{42}$$

Here we must note that in case of the even integer J the first term on the right hand side of Eq. (39) does not appear and so in such case we must set $l_{2J}^{lm}=0$. Then the final form of the synthetic kernel G_J^{lm} is the following :

$$G_J^{lm}(u'\to u) = r_J^{lm}\delta(u-u') + a_J^{lm}e^{-\lambda_J^{lm}(u-u')}$$
$$+\sum_{\mu=1}^{\left[\frac{J}{2}\right]} (-1)^{\mu} b_{J,\mu}^{lm} \cos\{\gamma_{J,\mu}^{lm}(u-u') - \theta_{J,\mu}^{lm}\}\, e^{-\beta_{J,\mu}^{lm}(u-u')}, \tag{43}$$

where

$$a_J^{lm} = l_{2J}^{lm} \cdot v_{2J}^{lm}\ ;\ b_{J,\mu}^{lm} = 2v_{2J,\mu}^{lm}\sqrt{(m_{2J,\mu}^{lm})^2+(n_{2J,\mu}^{lm})^2}\ ;\ \theta_{J,\mu}^{lm} = \tan^{-1}\frac{n_{2J,\mu}^{lm}}{m_{2J,\mu}^{lm}}\ . \tag{44}$$

The generalized Greuling Goertzel kernel for the l'th-order Legendre component of the elastic scattering transition probability is expressed in terms of the above G_J^{lm} kernel, i. e.

$$G_J^{l}(u'\to u) = \sum_{m=max(0,l-2J)}^{l+2J} f_m(u)\, G_J^{lm}(u'\to u) \tag{45}$$

The upper limit of this summation comes from the fact that since, as is known from Eq. (5), the smaller lethargy moments than T_{ll}^{2J} which is of the order $(1/M)^{2J}$ do not be taken into account in the G_{2J}^{ll} kernel, it is meaningless to include the G_{2J}^{lm} kernels which are constracted only by lethargy moments of the order smaller than $(1/M)^{2J}$.

B) $n = 2J-1$

In this case the polynomial $P_{2J-1}^{lm}(\mathbf{D})$ is of degree J, while $Q_{2J-1}^{lm}(\mathbf{D})$ is the polynomial of degree $J-1$. Then Equation (14) is reformulated in the form

$$\sum_{k=0}^{\infty} T_{lm}^{k}\, \mathbf{D}^k \cong v_{2J-1}^{lm}\, R_{2J-1}^{lm}(\mathbf{D}), \tag{46}$$

where

$$R_{2J-1}^{lm}(\mathbf{D}) = \frac{\sum_{\lambda=0}^{J-1} \sigma_{2J-1,\lambda}^{lm}\, \mathbf{D}^{\lambda}}{\sum_{\lambda=0}^{J} \nu_{2J-1,\lambda}^{lm}\, \mathbf{D}^{\lambda}}\ . \tag{47}$$

The coefficients v_{2J-1}^{lm}, $\sigma_{2J-1,\lambda}^{lm}$ and $\nu_{2J-1,\lambda}^{lm}$ are also determined in the similar procedure to the case of $n=2J$. In this case, however, these unknown parameters can easily be obtained by using $P_{2J-1,\lambda}^{lm}$ and $q_{2J-1,\lambda}^{lm}$ definedin Eqs. (24) and (25). That is,

$$v_{2J-1}^{lm} = \frac{q_{2J-1,J-1}^{lm}}{p_{2J-1,J}^{lm}} = (-1)^{J}\,\frac{D_{J+1,-1}^{lm}}{D_{J,1}^{lm}}\quad , \tag{48}$$

$$\nu_{2J-1,\lambda}^{lm} = \frac{p_{2J-1,\lambda}^{lm}}{p_{2J-1,J}^{lm}} = (-1)^J \frac{1}{D_{J,1}^{lm}} \cdot D_{J+1,0}^{lm}(\underbrace{0, 0, \cdots 0, 1}_{\lambda+1}, 0 \cdots 0), \tag{49}$$

$$\sigma_{2J-1,J}^{lm} = \frac{q_{2J-1,\lambda}^{lm}}{q_{2J-1,J-1}^{lm}} = \frac{1}{D_{J+1,-1}^{lm}} \cdot D_{J+1,0}^{lm}(T_{lm}^{\lambda}, T_{lm}^{\lambda-1}, \cdots T_{lm}^{0}, 0 \cdots 0), \tag{50}$$

where $T_{lm}^{-1}=0$.

Here were we can get the formal expression for another type of the generalized synthetic kernel, i. e.,

$$W_J^{lm}(u' \to u) = v_{2J-1}^{lm} \cdot R_{2J-1}^{lm}(\mathbf{D}) \cdot \delta(u-u'). \tag{51}$$

This is the generalization of the well-known Wigner kernel.

The similar procedure to the case of the G_{2J}^{lm} kernel give rise to the following explicit expression for the $W_J^{lm}(u' \to u)$:

$$W_J^{lm}(u' \to u) = c_J^{lm} e^{-\tau_J^{lm}(u-u')} + \sum_{\mu=0}^{\left[\frac{J}{2}\right]} (-1)^{\mu} d_{J,\mu}^{lm} \cos\{\eta_{J,\mu}^{lm}(u-u') - \varphi_{J,\mu}^{lm}\} e^{-\zeta_{J,\mu}^{lm}(u-u')}, \tag{52}$$

where the parameters c_J^{lm}, d_J^{lm}, τ_J^{lm}, $\zeta_{J,\mu}^{lm}$, $\eta_{J,\mu}^{lm}$ and $\varphi_{J,\mu}^{lm}$ correspond to a_J^{lm}, b_J^{lm}, λ_J^{lm}, $\beta_{J,\mu}^{lm}$, $\gamma_{J,\mu}^{lm}$ and $\theta_{J,\mu}^{lm}$ in G_J^{lm} kernel.

The form of this kernel resembles the $G_J^{lm}(u' \to u)$ very much except the delta term which appears in Eq. (43).

The generalized Wigner kernel for $f_l(u' \to u)$ has the form

$$W_J^l(u' \to u) = \sum_{m=max(0,\,l-2J+1)}^{l+2J-1} f_m(u)\, W_J^{lm}(u' \to u) \tag{53}$$

The upper limit of the summation in this equation is owing to the same circumstance as in the case of the generalized Greuling Goertzel kernel.

V. CONCLUDING REMARKS

Up to 1968, the synthetic kernel approximations (SKA) for neutron slowing down were mainly developed within the Greuling Goertzel approximation. The SKA method has a large merit that it enables us to allow a rapid evaluation of the collision integral.

It is obvious that the previous SKA's have two difficulties, namely a) they have a possibility that they give unphysical results in calculating the higher order Legendre component of the angular neutron flux, and b) they cannot demonstrate the detailed structure of neutron flux in the vicinity of the scattering resonance.

In this paper, keeping the previous SKA's merit, we have developed the SKA method in order to avoid the previous SKA's defects mentioned above. As a result we have obtained two generalized synthetic kernels, i. e., $G_n^{lm}(u' \to u)$ and $W_n^{lm}(u' \to u)$.

In the case of the isotropic scattering in the center of mass system the G_n^{oo} kernel have already been proposed as the G_n kernel by YAMAMURA and SEKIYA[13]. It must be noted that the previous G_n kernel corresponds to the G^{oo}_{2n+1} kernel of the present generalized Greuling Goertzel kernel and so does not include the G^{oo}_{2n} kernel.

As a matter of cource, these approaches developed here do not give analytical formula for neutron flux, but the G^{lm}_n and W^{lm}_n approximations are very useful for calculations of the collision integrals and the slowing parameters defined by DUNN and BECKER[(3)] and STACEY [(1)(6)]. Namely the collision integrals and the modified slowing down parameters can rapidly be evaluated with the help of simple recursion relations corresponding to G^{lm}_n and W^{lm}_n kernels.

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