On the Numerical Analysis of Elliptic Boundary Value Problems by the Successive Over relaxation Method

(Report 1)

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Summary. The successive over relaxation method is an effective iterative method for solving the difference analogue of an elliptic partial differential equation. But its main difficulty is that the optimum relaxation factor for any given problem can not usually be found prior to getting the solution.

We[1,2] have proposed an empirical new method, practical successive over relaxation method, for solving the Dirichlet problem. This computational method was named NIKI's method by Radlley et al.[3]. In this paper, we describe the proposed method and report the results of it and two other methods of successive over relaxation method by applying them to same problems.

Introduction.

One of the numerical methods solving an elliptic partial differential equation is the successive over relaxation method, S. O. R.. In solving elliptic differential equations by finite difference approximation, it is frequently necessary to solve a set of linear equaitons, which can be expressed in the form

$$\boldsymbol{A}\boldsymbol{X}=\boldsymbol{b} \tag{1}$$

where $\boldsymbol{A}$ is a symmetric, positive deffinite matrix of order N with ones on the diagonal, and $\boldsymbol{X}$ and $\boldsymbol{b}$ are column vectors. If we define $\boldsymbol{L}$ as the lower triangular matrix of $\boldsymbol{A}$ with zeros on the diagonal, then $\boldsymbol{L}'$, the transpose of $\boldsymbol{L}$, is the upper trianglar matrix of $\boldsymbol{A}$. In terms of these matrices the iterative process of S. O. R. is defined by following relation;

$$\boldsymbol{X}^{(k)}=\boldsymbol{M}\boldsymbol{X}^{(k-1)}+\boldsymbol{C} \qquad (k=1, 2, \cdots) \tag{2}$$

where

$$\boldsymbol{M}=-(\omega\boldsymbol{L}+\boldsymbol{I})^{-1}[\omega\boldsymbol{L}'+(\omega-1)\boldsymbol{I}] \tag{3}$$

and

$$\boldsymbol{C}=(\omega^{-1}\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{b} \tag{4}$$

and ω is a parameter known as the accelerating factor.
If we now define an error vector as $\boldsymbol{e}^{(k)}=\boldsymbol{X}^{(k)}-\boldsymbol{X}$, where $\boldsymbol{X}$ is the exact solution $\boldsymbol{A}^{-1}\boldsymbol{b}$, it follows from equations (2) to (4) that

$$\boldsymbol{e}^{(k)}=\boldsymbol{M}\boldsymbol{e}^{(k-1)}.$$

To converge the iterative process, the condition

$$\lim_{k\to\infty}\boldsymbol{e}^{(k)}=0$$

holds for any arbitrary initial vector $\boldsymbol{e}^{(0)}$, so that it is necessary for all the eigenvalues of $\boldsymbol{M}$ to lie within the unit circle. When the parameter ω takes the value of unity, the iterative

process defined above is the classical Gauss-Seidel method. The object of using a value of ω other than unity is to reduce the spectral radius, i. e. the largest absolute values of the eigenvalues of the matrix $\boldsymbol{M}$ (We shall denote this spectral radius by λ_{max}).

Ideally one would use the value known as the optimum accelerating factor, ω_0, which minimizes λ_{max}.

Young[4] has shown that the optimum accelerating factor is given by

$$\omega=2/\{1+\sqrt{(1-\lambda_{max})}\}.$$

It is, however, not easy to obtain the optimum ω for complex regions. Carré[5] and Forsythe[6] have respectively proposed methods of S. O. R. in which successively better estimates of the optimum accelerating factor are obtained during the course of solution, but their methods are very time-consuming.

We have empirically noticed that, after several scans have been performed using ω smaller than the optimum value, all $\boldsymbol{E}_{i,j}^{k}$ are positive; when the factor is larger than the optimum value, however, some $\boldsymbol{E}_{i,j}^{k}$ are found to be negative. (The symbol $\boldsymbol{E}_{i,j}^{k}$ denotes the residue at each mesh point in scanning.)

This property appears after several scans. The number of scans for which all $\boldsymbol{E}_{i,j}^{k}$ are positive is dependent on the mesh to be caluculated.

For instance, on 100 mesh points, negative signs appear after five scans; on 1000 mesh points, they appear after ten scans. When the acceleration factor is smaller than the optimum value, however, negative signs are only a few percent of the $\boldsymbol{E}_{i,j}^{k}$.

Whereas, when the acceleration factor is larger than the optimum value, many negative signs appear even after a few scans, Using this property, we have developed a new technique for the adaptive selection of approximate accelerating factor. The accelerating factor is varied so that it tends asymptotically to its optimum value. The proposed method is very simple compared to others, especially that of Carré, in view of the procedure for calculation.

The procedure is as follows. At first, all interior mesh points are set to be zero initially and the accelerating factor ω is set up arbitrarily (it is here set to 1.5 for convenience). Using this value, iterations are performed several times and the signs of all $\boldsymbol{E}_{i,j}^{k}$ are descriminated thereafter.

(a). If they are all positive, 0.1 is added to ω. In this case only a few percent of the negative signs are discorded. If all the signs of the $\boldsymbol{E}_{i,j}^{k}$ are positive when several iterations and discriminations have been performed with this ω, the same procedure is repeated until a negative sign appears. Then 0.05 is subtracted from ω, and, adopting this value as the approximate accelerating factor, iterations are repeated until all the $\boldsymbol{E}_{i,j}^{k}$ converge.

(b). If some $\boldsymbol{E}_{i,j}^{k}$ have negative signs even at the first descrimination, 0.1 is subtracted from ω. When several iterations and discriminations have been performed with this ω; if a negative sign appears, the same procedure is repeated until signs of all $\boldsymbol{E}_{i,j}^{k}$ become positive. Then 0.05 is added to ω, and this value is adopted as the approximate accelerating factor, iterations being repeated until all the $\boldsymbol{E}_{i,j}^{k}$ converge. It has been found that the sign

property of the $E_{i,j}^{k}$ can be definitely found it more than four iterations are taken.
We set the iteration number at 5 so that ω approaches the optimum accelerating factor as soon as possible.

Experimental Result.

One of the FORTRAN program lists coding the proposed method is shown in Fig. (1). We tried the Laplace's equation, using square mesh and the usual five-points difference formula, to the Dirichlet problem,
In all cases, we judged a solution to be obtained when all $E_{i,j}^{k}$ were equal to, or less

Table. 1 (case A) Comparison of Proposed and Other Methods

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
13×13	29	29	38	38	47	47
17×17	44	44	×	×	66	66
23×23	54	54	67	68	96	96
25×25	56	56	×	×	103	103
30×30	62	62	×	×	129	129
32×32	68	68	76	77	138	138
35×35	66	66	80	92	154	154
37×37	70	70	83	×	165	165

N = iteration number for convergence.
× = no convergence.

Table. 1 (case B) Comparison of Proposed and Other Methods

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
20×15	43	43	×	×	65	65
20×25	54	54	57	×	87	87
15×40	46	46	58	59	78	78
20×35	59	59	66	66	99	99
20×40	61	61	68	×	102	102
25×33	63	63	65	81	118	118
21×43	62	62	73	73	111	111
25×40	60	60	83	86	127	127

N =iteration number for convergence.
× = no convergence.

than 10^{-6}.

In case A, various size of square are used, and in case B, some rectangular regions.

In case 1, the scanning direction is Y-direction in two-dimensional Cartesian co-ordinates and r-direction in axially symmetric co-ordinate.

In case 2, contrary, the scanning direction is X-directon and Z-diredtion.

Table 1 gives the interation numbers of the proposed method and compares them with other methods in two-dimensional Cartesian co-ordinates.

Table 2 gives the results in axially symmetric co-ordinates (similar to Table 1).

Next, we tried the methods on mixed boundary problem.

Table. 2 (case A) Comparison of Proposed and Other Methods.

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
15×15	44	44	×	×	59	59
20×20	59	59	64	64	80	80
23×23	64	64	×	×	97	97
25×25	64	64	×	×	110	110
30×30	70	70	×	×	134	134
32×32	70	70	×	×	142	142
35×35	83	83	×	×	159	159
40×40	103	103	×	272	186	186

N = iteration number for convergence.
× = no convergence.

Table. 2 (case B) Comparison of Proposed and Other Methods

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
10×15	24	24	37	37	39	39
16×25	48	48	55	55	76	76
20×25	63	63	×	×	92	92
15×40	53	53	61	61	82	82
20×40	67	67	68	69	107	107
25×33	67	67	×	×	124	124
21×43	70	70	74	74	117	117
25×40	67	67	87	88	134	134

N = iteration number for convergence.
× = no convergence

In case A, Neumann conditions were specified on one side of the square and Dirichlet conditions on the others. In case B, Neuman conditions were applied to the longer side and Dirichlet conditions to the other three sides.

Table 3 gives the results in 2-dimensional Cartesian co-ordinates (similar to Table 1).

Table 4 gives the results in axially symmetric co-ordinates (similar to Table 2).

Using Carré's method in some problems, the solution never converged, owing to the occurrence of a complex value in the over relaxation parameter.

In case of Forsythe's method, the solution converged in all problems. However, his method was very time-consuming.

Table. 3 (case A) Comparison of Proposed and Other Methods

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
13×13	40	40	48	48	59	59
17×17	50	50	65	65	79	79
23×23	61	61	74	75	116	116
25×25	63	63	×	×	129	129
30×30	69	69	75	83	159	159
32×32	80	80	×	94	172	172
35×35	96	96	99	102	191	191
37×37	106	106	×	×	205	205

N = iteration number for convergence.
× = no convergence.

Table. 3 (case B) Comparison of Proposed and Other Methods.

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
20×15	52	52	61	61	93	93
20×25	58	58	64	65	100	100
15×40	50	50	×	×	78	78
20×35	61	61	64	65	103	103
25×33	63	63	85	73	136	136
21×43	60	60	72	72	115	115
25×40	67	67	85	84	139	139
20×40	60	60	62	62	108	108

N = iteration number for convergence.
× = no convergence.

Table. 4 (case A) Comparison of Proposed and Other Methods

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
15×15	45	45	×	×	62	62
20×20	60	60	×	×	94	94
23×23	66	66	×	×	100	100
25×25	70	70	×	×	116	116
30×30	76	76	×	×	139	139
32×32	83	83	×	×	154	154
35×35	90	90	122	118	165	165
40×40	118	118	×	×	197	197

N = iteration number for convergence.
× = no convergence.

Table. 4 (case B) Comparison of Proposed and Other Methods.

	Proposed method		Carré's method		Forsythe's method	
	N		N		N	
	case 1	case 2	case 1	case 2	case 1	case 2
10×15	31	31	40	40	41	41
16×25	51	51	57	57	8I	81
20×25	65	65	71	94	96	96
15×40	58	58	62	62	86	86
20×40	71	71	72	72	113	113
25×33	66	66	70	71	132	132
21×43	73	73	77	77	118	118
25×40	74	74	91	89	139	139

N = iteration number for convergence.

Conclusion.

In accordance with our experiments on mixed boundary problem, it is clear that the proposed method can be performed easily to estimate the over relaxation parameter and requires fewer iterations than the others.

Acknowlegement

We are very grateful Miss Akutsu of Okayama College of Science, for thier assistance with computer programming. We used a HITAC-8400-65 ccomputer.

Fig. 1 Sample program of the proposed method.

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1 C      PROPOSED METHOD FOR DIRICHLET BOUNDARY PROBLEM
2        INTER*2  L (50, 50)
3        DIMENSION U (50, 50)
4        READ(5, 99) IMAX, JMAX, ERS, H
5     99 FORMAT (2I3, 2F7. 4)
6 C      EPS-MAXIMUM RESIDUAL
7        READ (5, 100) L
8    100 FORMAT (50I1)
9        DO 10 I=1, IMAX
10       DO 10 J=1, JMAX
11       IF (L (I, J)-2) 70, 71, 70
12    70 U (I, J)=0. 0
13       GO TC 10
14    71 U(I, J)=1. 0
15    10 CONTINUE
16       K=0
17 C     K-ITERATION UNMBER WITH DECREASED OMEGA
18       OMEGA=1.5
19  2000 DO 16 M=1, 5
20       NEGATI=0
21 C     NEGATI-NUMBER OF NEGATIVE RESIDUAL
22       NUMBER=0
23 C     NUMBER-NUMBER OF CALCULATING MESH POINTS
24       NCONVE=0
25 C     NCCNVE-NUMBER OF COMBERGING MESH POINTS
26       DO 17 I=1, IMAX
27       DO 17 J=1, JMAX
28       IF (L (I, J) .LE. 2) GO TO 17
29       RESIDU OMFGA*((U(I-1, J)+U(I, J-1)+U(I+1, J)+U(I, J+1))/4. -U(I, J))(H*H)
30        U(I, J)=U(J, J)+RESIDU
31       NUMBER=NUMBER+1
32       ARESID=ABS (RESIUD)
33       IF (ARESID .LE. EPS) GO TO 18
34       NCONVE=NCONVE+1
35    18 IF (RESIDU .GE. 0.) GO TO 17
36       NEGATI=NEGATI+1
37    17 CONTINNUE
38       IF (NCONVE .EQ. 0) GO TO 5000
39    16 CONTINUE
40       IF (K .GE. 1) GO TO 2000
41       IF (NEGATI .LE. (NUMBER*0.05)) GO TO 1350
42       OMEGA=OMAGE-0.05
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43          K=K+1
44          GO TO 2000
45    1350 OMEGA=OMEGA+0.1
46          IF(OMEGA .GE. 1.8) OMEGA=1.85
47          GO TO 2000
48    5000 WRITE (6, 21) ((I, J, U(I, J), I=1, IMAX), J=1, JMAX)
49      21 FORMAT (3 (5X, I3, 5I3, 'U (I,J)='E15.7))
50          STOP
51          END
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