A Note on Kählerian Spaces with Vanishing Bochner Curvature Tensor

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§ 1. **Introduction.** An n (=2m) dimensional Kählerian space is a Riemannian space which admits a structure tensor $\varphi_\mu{}^\lambda$ satisfying

$$\varphi_\mu{}^\alpha \varphi_\alpha{}^\lambda = -\delta_\mu^\lambda ,$$
$$\varphi_{\lambda\mu} = -\varphi_{\mu\lambda} , \quad (\varphi_{\lambda\mu} = \varphi_\lambda{}^\alpha g_{\alpha\mu}) ,$$
$$\nabla_\mu \varphi_\lambda{}^\kappa = 0 .$$

where ∇_μ means the operator of covariant differentiation. It is well known that the holomorphically projective curvature tensor $P_{\lambda\mu\nu}{}^\kappa$ [1] of a Kählerian space, which is invariant under any holomorphically projective correspondence, corresponds to the Weyl's projective curvature tensor $W_{\lambda\mu\nu}{}^\kappa$ of a Riemannian space, which is invariant under any projective correspondence. On the other hand, in a Kählerian space S. Bochner has introduced a tensor $K_{\alpha\bar\beta\gamma\bar\delta}$ [2] with respect to complex local coordinates, which is the formal analogy of the Weyl's conformal curvature tensor of a Riemannian space.

Recently S. Tachibana has showed that with respect to real local coordinates a tensor $K_{\lambda\mu\nu\omega} = K_{\lambda\mu\nu}{}^\kappa g_{\kappa\omega}$ [3] defined by

$$\begin{aligned} K_{\lambda\mu\nu}{}^\kappa = R_{\lambda\mu\nu}{}^\kappa + \frac{1}{n+4}(R_{\lambda\nu}\delta_\mu^\kappa - K_{\mu\nu}\delta_\lambda^\kappa + g_{\lambda\nu}R_\mu{}^\kappa - g_{\mu\nu}R_\lambda{}^\kappa + S_{\lambda\nu}\varphi_\mu{}^\kappa - S_{\mu\nu}\varphi_\lambda{}^\kappa \\ + \varphi_{\lambda\nu}S_\mu{}^\kappa - \varphi_{\mu\nu}S_\lambda{}^\kappa + 2S_{\lambda\mu}\varphi_\nu{}^\kappa + 2\varphi_{\lambda\mu}S_\nu{}^\kappa) \\ - \frac{R}{(n+2)(n+4)}(g_{\lambda\nu}\delta_\mu^\kappa - g_{\mu\nu}\delta_\lambda^\kappa + \varphi_{\lambda\nu}\varphi_\mu{}^\kappa - \varphi_{\mu\nu}\varphi_\lambda{}^\kappa + 2\varphi_{\lambda\mu}\varphi_\nu{}^\kappa) ,\end{aligned}$$

where $S_{\mu\nu} = \varphi_\mu{}^\alpha R_{\alpha\nu}$, has components of the tensor given by S. Bochner, and has called this tensor the Bochner curvature tensor [2]. In his paper the next theorem has been proved.

Theorem 1. (S. Tachibana) If a compact Kählerian space with vanishing Bochner curvature tensor of constant scalar curvature has positive definite Ricci form, then it is a complex projective space with the natural metric.

On the other hand, in a previous paper [1] the present author has proved the following

Theorem 2. In a compact Kählerian space with vanishing Bochner curvature tensor, $g^{\lambda\mu}\nabla_\mu R$ *is a contravariant analytic vector.*

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1) K. Yano, [5] p. 265, Y. Tashiro, [3].

2) K. Yano and S. Bochner, [4] p. 162.

3) As to notations we follow M. Matsumoto, [1], S. Tachibana, [2].

In this paper we shall prove the following

Theorem 3. In a compact Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvatnre and positive definite Ricci form, if $R_{\mu\lambda}R^{\mu\lambda}$ *is constant, then it is a complex projective space with natural metric.*

§ 2. **Vanishing Bochner curvature tensor.** In a Kählerian space the next equations hold good.

$$(2.1)\quad \begin{cases} R_{\alpha\mu\nu}{}^{\kappa}\varphi_{\lambda}{}^{\alpha}=-R_{\lambda\alpha\nu}{}^{\kappa}\varphi_{\mu}{}^{\alpha}\ ,\quad R_{\lambda\mu\alpha}{}^{\kappa}\varphi_{\nu}{}^{\alpha}=R_{\lambda\mu\nu}{}^{\alpha}\varphi_{\alpha}{}^{\kappa}\ , \\ \varphi_{\lambda}{}^{\alpha}R_{\alpha\mu}=-R_{\lambda\alpha}\varphi_{\mu}{}^{\alpha}\ ,\quad \varphi_{\lambda}{}^{\alpha}R_{\alpha}{}^{\kappa}=R_{\lambda}{}^{\alpha}\varphi_{\alpha}{}^{\kappa}\ , \\ \nabla_{\alpha}R_{\lambda\mu\nu}{}^{\alpha}=\nabla_{\lambda}R_{\mu\nu}-\nabla_{\mu}R_{\lambda\nu}\ ,\quad \nabla_{\lambda}R=2\nabla_{\alpha}R_{\lambda}{}^{\alpha}\ . \end{cases}$$

It is known that the tensor $S_{\mu\nu}$ defined by

$$S_{\mu\nu}=\varphi_{\mu}{}^{\alpha}R_{\alpha\nu}$$

satisfies the following equations:

$$(2.2)\quad \begin{cases} S_{\mu\nu}=-S_{\nu\mu}\ ,\quad 2\nabla_{\alpha}S_{\lambda}{}^{\alpha}=\varphi_{\lambda}{}^{\alpha}\nabla_{\alpha}R\ , \\ \varphi_{\lambda}{}^{\alpha}S_{\alpha\nu}=-S_{\lambda\alpha}\varphi_{\nu}{}^{\alpha}=-R_{\lambda\nu}\ , \\ S_{\mu\nu}=-\frac{1}{2}\varphi^{\alpha\beta}R_{\mu\nu\alpha\beta}=\varphi^{\alpha\beta}R_{\alpha\mu\nu\beta}\ . \end{cases}$$

As the differential form $S=\frac{1}{2}S_{\mu\lambda}dx^{\mu}{}_{\wedge}dx^{\lambda}$ is closed [4], it follows that

$$(2.3)\qquad \varphi_{\nu}{}^{\alpha}\nabla_{\alpha}S_{\mu\lambda}=\nabla_{\lambda}R_{\mu\nu}-\nabla_{\mu}R_{\lambda\nu}\ .$$

Transvecting (2.3) with $\varphi_{\sigma}{}^{\nu}\varphi_{\rho}{}^{\mu}$, we obtain

$$(2.4)\qquad \nabla_{\sigma}R_{\rho\lambda}=\varphi_{\sigma}{}^{\nu}\varphi_{\rho}{}^{\mu}\,(\nabla_{\lambda}R_{\mu\nu}-\nabla_{\mu}R_{\lambda\nu})\ .$$

In a Kählerian space with vanishing Bochner curvature tensor the equation $\nabla_{\omega}K_{\lambda\mu\nu}{}^{\kappa}=0$ holds good, and we have

$$\begin{aligned}(2.5)\qquad \nabla_{\omega}R_{\lambda\mu\nu}{}^{\kappa}+\frac{1}{n+4}[&(\nabla_{\omega}R_{\lambda\nu})\delta_{\mu}^{\kappa}-(\nabla_{\omega}R_{\mu\nu})\delta_{\lambda}^{\kappa}+g_{\lambda\nu}\nabla_{\omega}R_{\mu}{}^{\kappa}-g_{\mu\nu}\nabla_{\omega}R_{\lambda}{}^{\kappa} \\ &+(\nabla_{\omega}S_{\lambda\nu})\varphi_{\mu}{}^{\kappa}-(\nabla_{\omega}S_{\mu\nu})\varphi_{\lambda}{}^{\kappa}+\varphi_{\lambda\nu}\nabla_{\omega}S_{\mu}{}^{\kappa}-\varphi_{\mu\nu}\nabla_{\omega}S_{\lambda}{}^{\kappa} \\ &+2(\nabla_{\omega}S_{\lambda\mu})\varphi_{\nu}{}^{\kappa}+2\varphi_{\lambda\mu}\nabla_{\omega}S_{\nu}{}^{\kappa}] \\ -\frac{\nabla_{\omega}R}{(n+2)(n+4)}&(g_{\lambda\nu}\delta_{\mu}^{\kappa}-g_{\mu\nu}\delta_{\lambda}^{\kappa}+\varphi_{\lambda\nu}\varphi_{\mu}{}^{\kappa}-\varphi_{\mu\nu}\varphi_{\lambda}{}^{\kappa}+2\varphi_{\lambda\mu}\varphi_{\nu}{}^{\kappa})=0\ .\end{aligned}$$

Contracting (2.5) with respect to κ and ω and making use of (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned}\nabla_{\lambda}R_{\mu\nu}-\nabla_{\mu}R_{\lambda\nu}+\frac{1}{2(n+2)}(&g_{\lambda\nu}\nabla_{\mu}R-g_{\mu\nu}\nabla_{\lambda}R+\varphi_{\lambda\nu}\varphi_{\mu}{}^{\alpha}\nabla_{\alpha}R \\ &-\varphi_{\mu\nu}\varphi_{\lambda}{}^{\alpha}\nabla_{\alpha}R+2\varphi_{\lambda\mu}\varphi_{\nu}{}^{\alpha}\nabla_{\alpha}R)=0\ .\end{aligned}$$

Transvecting the above equation with $\varphi_{\sigma}{}^{\nu}\varphi_{\rho}{}^{\mu}$ and by virtue of (2.4), we find

4) K. Yano, [5] p. 72.

$$(2.6)\qquad \nabla_\nu R_{\mu\lambda}=\frac{1}{2(n+2)}(g_{\nu\mu}\nabla_\lambda R+g_{\nu\lambda}\nabla_\mu R-\varphi_{\nu\mu}\varphi_\lambda{}^\alpha\nabla_\alpha R-\varphi_{\nu\lambda}\varphi_\mu{}^\alpha\nabla_\alpha R+2g_{\mu\lambda}\nabla_\nu R)\ .$$

Thus if the scalar curvature R is constant, then we have $\nabla_\nu R_{\mu\lambda}=0$. Hence by virtue of (2.5) it follows that the Kählerian space is symmetric. Thus we have

Lemma 2.1. If a Kählerian space with vanishing Bochner curvature tensor has constant scalar curvature, then it is a symmetric space.

If we put $u_\lambda=\nabla_\lambda R$, then (2.6) becomes the following form:

$$(2.7)\qquad \nabla_\nu R_{\mu\lambda}=\frac{1}{2(n+2)}(g_{\nu\mu}u_\lambda+g_{\nu\lambda}u_\mu-\varphi_{\nu\mu}\varphi_\lambda{}^\alpha u_\alpha-\varphi_{\nu\lambda}\varphi_\mu{}^\alpha u_\alpha+2g_{\mu\lambda}u_\nu)$$

Transvecting (2.7) with $R^{\mu\lambda}$, we obtain

$$\nabla_\alpha(R_{\mu\lambda}R^{\mu\lambda})=\frac{2}{n+2}(2R_{\beta\alpha}u^\beta+Ru_\alpha)\ .$$

If $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then it follows that

$$(2.8)\qquad 2R_{\alpha\beta}u^\beta+Ru_\alpha=0\ .$$

Transvecting (2.8) with u^α , we have

$$2R_{\beta\alpha}u^\beta u^\alpha+Ru_\alpha u^\alpha=0\ .$$

If a Kählerian space with vanishing Bochner curvature tensor has positive definite Ricci form and non-negative scalar curvature, then it follows that $u^\lambda=g^{\lambda\mu}\nabla_\mu R=0$. Thus we have

Lemma 2.2. In a Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvature and positive definite Ricci form, if $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then the scalar curvature is constant.

According to Lemma 2.1 we obtain

Proposition 2.3. In a Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvature and positive definite Ricci form, if $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then the space is symmetric.

By virtue of Theorem 2, $g^{\lambda\mu}\nabla_\mu R$ is contravariant analytic in a compact Kählerian space with vanishing Bochner curvature tensor. Thus the equation $\nabla^\alpha\nabla_\alpha u^\lambda+R_\alpha{}^\lambda u^\alpha=0$ [5] holds good. If $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then substituting this equation into (2·8) we get

$$(2.9)\qquad 2\nabla^\alpha\nabla_\alpha u^\lambda=Ru^\lambda\ .$$

On the other hand, operating the Laplacian $g^{\beta\alpha}\nabla_\beta\nabla_\alpha$ to $u^\lambda u_\lambda$, we have

$$\nabla^\mu\nabla_\mu(u^\lambda u_\lambda)=2(\nabla^\mu\nabla_\mu u^\lambda)u_\lambda+2(\nabla_\mu u_\lambda)(\nabla^\mu u^\lambda)\ .$$

Substituting (2.9) into the above equation, we obtain

$$\nabla^\mu\nabla_\mu(u^\lambda u_\lambda)=Ru_\lambda u^\lambda+2(\nabla_\mu u_\lambda)(\nabla^\mu u^\lambda)\ .$$

5) K. Yano, [5] p. 86.

If our space has the non-negative scalar curvature, then we find $u^{\lambda}=g^{\lambda\mu}\nabla_{\mu}R=0$ taking account of Green's theorem. Thus we have

Lemma 2.4. In a compact Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvature, if $R_{\mu\lambda}R^{\mu\lambda}$ is cvnstant, then the scalar curvature is constant.

By virtue of Lemma 2.1, we obtain

Proposition 2.5. In a compact Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvature, if $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then the space is symmetric.

Proof of Theorem 3. Taking account of Lemma 2.2 or Lemma 2.4, in a compact Kählerian space with vanishing Bochner curvature tensor of non-negative scalar curvature and positive definite Ricci form, if $R_{\mu\lambda}R^{\mu\lambda}$ is constant, then the scalar curvature is constant. According to Theorem 1, it follows that the space is a complex projective one with natural metric. Q. E. D.

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