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Hyers–Ulam–Rassias stability of first-order homogeneous linear difference equations with a small step size

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Abstract. This paper deals with the stability in the sense of Hyers–Ulam–Rassias of constant step size linear difference equation $\Delta_h x(t) - ax(t) = 0$, where $a \in \mathbb{R}$. In particular, we establish stability results of this equation under the assumption that $h > 0$ is small step size: $a \neq 0$ and $a > -1/h$.

Keywords: Hyers–Ulam–Rassias stability; Hyers–Ulam stability; linear difference equation; constant step size; small step size.

1 Introduction

The problem of Hyers–Ulam stability (HUS) was posed by Ulam [25] in 1940. A year later, Hyers [10] gave a partial answer to this problem. After that, it has been investigated and generalized by many researchers. It is well known that Hyers–Ulam–Rassias stability (HURS) is a generalization of HUS. These problems were first addressed in the field of functional equations. The reader see the books written by Brzdęk, Popa, Raşa and Xu [8], and Jung [11] for historical backgrounds on Hyers–Ulam stability and Hyers–Ulam–Rassias stability. For recent references on HURS of functional equations, see [12, 26]. In 1998, Alsina and Ger [1] studied HUS for the simple differential equation

$$x' - x = 0.$$

This study has been improved and extended recently, and is ongoing (see, [18, 23]). In addition, this problem has spread to the other areas. For example, we can find it in the field of difference equations, dynamic equations on time scales, delay differential equations (see, [2, 3, 4, 6, 7, 15, 16, 17, 20, 21, 22]). In recent years, Hyers–Ulam–Rassias stability has also been considered in the field of differential equations. For example, see [5, 9, 13, 14, 19, 21, 24].

It is well known that the derivative x' can be approximated by the following difference:

$$\Delta_h x(t) := \frac{x(t+h) - x(t)}{h}$$

for $h > 0$. $\Delta_h x(t)$ and $h > 0$ are so called the "*forward difference operator*" and the "*step size*". Define

$$h\mathbb{Z} := \{hk|\, k \in \mathbb{Z}\}$$

for $h > 0$. In 2017, Onitsuka [16] studied HUS of the first-order homogeneous linear difference equation

$$\Delta_h x(t) - ax(t) = 0, \quad t \in h\mathbb{Z}, \tag{1.1}$$

where $a \in \mathbb{R} \setminus \{-1/h\}$. Note that we no longer have a difference equation when $a = -1/h$. For this reason, we assume $a \neq -1/h$. Before giving a definition of the stability, we will give some sets. Let I be a nonempty open interval of $\mathbb{R}$, and let $\mathbb{T} := h\mathbb{Z} \cap I$. If the maximum of $\mathbb{T}$ exists, define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\max \mathbb{T}\}$; otherwise, $\mathbb{T}^\kappa := \mathbb{T}$. Now we will give a definition of stability. (1.1) is "*Hyers–Ulam stable (HUS)*" on $\mathbb{T}$ if and only if there exists a constant $K > 0$ such that the following holds:

Let $\varepsilon > 0$ be a given constant. If for any function $\xi : \mathbb{T} \to \mathbb{R}$ satisfying $|\Delta_h \xi(t) - a\xi(t)| \leq \varepsilon$ for all $t \in \mathbb{T}^\kappa$, there exists a solution $x : \mathbb{T} \to \mathbb{R}$ of (1.1) such that $|\xi(t) - x(t)| \leq K\varepsilon$ for all $t \in \mathbb{T}$.

In [16], the following result was given.

Theorem A ([16], Theorem 1.5). *Suppose that $a \neq 0$ and $a > -1/h$. Let $\varepsilon > 0$ be a given constant. Suppose also that a function $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h \xi(t) - a\xi(t)| \leq \varepsilon$ for all $t \in \mathbb{T}^\kappa$. Then the following hold:*

(i) if $a > 0$ and $\bar{t} := \max \mathbb{T}$ exists, then any solution $x(t)$ of (1.1) with $|\xi(\bar{t}) - x(\bar{t})| < \varepsilon/a$ satisfies that $|\xi(t) - x(t)| < \varepsilon/a$ for all $t \in \mathbb{T}$;

(ii) if $a > 0$ and $\max \mathbb{T}$ does not exist, then $\lim_{t\to\infty} \xi(t)(ah+1)^{-t/h}$ exists, and there exists a unique solution

$$x(t) = \left\{ \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

of (1.1) such that $|\xi(t) - x(t)| \leq \varepsilon/a$ for all $t \in \mathbb{T}$;

(iii) if $-1/h < a < 0$ and $\underline{t} := \min \mathbb{T}$ exists, then any solution $x(t)$ of (1.1) with $|\xi(\underline{t}) - x(\underline{t})| < \varepsilon/|a|$ satisfies that $|\xi(t) - x(t)| < \varepsilon/|a|$ for all $t \in \mathbb{T}$;

(iv) if $-1/h < a < 0$ and $\min \mathbb{T}$ does not exist, then $\lim_{t\to-\infty} \xi(t)(ah+1)^{-t/h}$ exists, and there exists a unique solution

$$x(t) = \left\{ \lim_{t\to-\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

of (1.1) such that $|\xi(t) - x(t)| \leq \varepsilon/|a|$ for all $t \in \mathbb{T}$.

This result implies the following.

Corollary B ([16], Corollary 4.1). *If $a \neq 0$ and $a > -1/h$, then (1.1) is HUS on $\mathbb{T}$.*

If the step size $h > 0$ is sufficiently small, then $a \neq 0$ and $a > -1/h$ hold. We call this case "*small step size*" case. Note here that Theorem A and Corollary B were obtained under the assumption that $h > 0$ is small step size, but the other cases were also discussed in [16]. Recently, the results obtained in [16] have already been extended to various general equations (see, [2, 3, 4, 17]). Moreover, (1.1) corresponds to the differential equation $x' - ax = 0$ when $h \to 0$, so that, (1.1) is called an approximate equation of the differential equation $x' - ax = 0$. HUS of $x' - ax = 0$ was studied by Onitsuka and Shoji [18] in 2017. Namely, small step size case means that an approximation problem for differential equations.

The purpose of this study is to extend the results obtained above to more general stability results. (1.1) is said to be "*Hyers–Ulam–Rassias stable (HURS)*" or "*Aoki–Rassias stable*" on $\mathbb{T}$ if allowing $\varepsilon > 0$ and $K > 0$ in HUS to depend on $t \in h\mathbb{Z}$, that is, if there exists a positive function $\psi : \mathbb{T} \to \mathbb{R}$ such that the following holds:

Let $\phi : \mathbb{T}^\kappa \to \mathbb{R}$ be a given positive function. If for any function $\xi : \mathbb{T} \to \mathbb{R}$ satisfying $|\Delta_h \xi(t) - a\xi(t)| \leq \phi(t)$ for all $t \in \mathbb{T}^\kappa$, there exists a solution $x : \mathbb{T} \to \mathbb{R}$ of (1.1) such that $|\xi(t) - x(t)| \leq \psi(t)$ for all $t \in \mathbb{T}$.

In the next section, we will discuss some properties for nonhomogeneous linear difference equations. In section 3, we will give the main theorem and its proof.

2 Nonhomogeneous linear difference equations

In this section, we present some properties of the solutions of the first-order nonhomogeneous linear difference equation

$$\Delta_h x(t) - ax(t) = f(t) \tag{2.1}$$

on $h\mathbb{Z}$, where a is a real number and $f(t)$ is a real-valued function on $h\mathbb{Z}$. Let

$$F(t; t_0, \Phi_0, f) := \begin{cases} \Phi_0 + h \displaystyle\sum_{i=1}^{(t-t_0)/h} f(t_0 + (i-1)h)(ah+1)^{-\frac{t_0}{h}-i}, & \text{if} \quad t \ge t_0 + h, \\ \Phi_0, & \text{if} \quad t = t_0, \\ \Phi_0 - h \displaystyle\sum_{i=1}^{(t_0-t)/h} f(t_0 - ih)(ah+1)^{-\frac{t_0}{h}+i-1}, & \text{if} \quad t \le t_0 - h \end{cases} \tag{2.2}$$

for $t \in h\mathbb{Z}$, where $t_0 \in h\mathbb{Z}$ and $\Phi_0 \in \mathbb{R}$ are arbitrary constants. Then the following is known (see [17, Lemma 3.3]).

Lemma 2.1. *Let $t_0 \in h\mathbb{Z}$ and $x_0 \in \mathbb{R}$. If $a \ne -1/h$ then the solution of the initial-value problem* (2.1) *with $x(t_0) = x_0$ is*

$$x(t) = F\left(t; t_0, x_0(ah+1)^{-\frac{t_0}{h}}, f\right)(ah+1)^{\frac{t}{h}}$$

for $t \in h\mathbb{Z}$, where F is the function given by (2.2).

Lemma 2.2. *Suppose that $a \ne 0$ and $a > -1/h$, and there exists an $L > 0$ such that $0 < f(t) \le L$ for all $t \in h\mathbb{Z}$. Let $t_0 \in h\mathbb{Z}$ and $x_0 \in \mathbb{R}$. Then $F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right)$ is an increasing function on $h\mathbb{Z}$, where F is the function given by* (2.2), *and the following hold:*

(i) if $a > 0$ and $x_0 \le -L/a$, then $\lim_{t\to\infty} F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right)$ exists and

$$F\left(t; t_0, x_0(ah+1)^{-\frac{t_0}{h}}, f\right) < \lim_{t\to\infty} F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right) \le 0$$

for all $t \in h\mathbb{Z}$;

(ii) if $-1/h < a < 0$ and $x_0 \ge -L/a$, then $\lim_{t\to-\infty} F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right)$ exists and

$$F\left(t; t_0, x_0(ah+1)^{-\frac{t_0}{h}}, f\right) > \lim_{t\to-\infty} F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right) \ge 0$$

for all $t \in h\mathbb{Z}$.

Proof. For the simplicity, let $G(t) := F\left(t; t_0, x_0(ah+1)^{-t_0/h}, f\right)$ on $h\mathbb{Z}$. Since $ah+1 > 0$ and $f(t)$ is a positive function, $G(t)$ is an increasing function on $h\mathbb{Z}$.

First we prove the case $a > 0$ and $x_0 \le -L/a$. Since $G(t)$ is increasing on $h\mathbb{Z}$, we have only to prove that $G(t) < 0$ for all $t \ge t_0 + h$. Using $a > 0$ and $x_0 \le -L/a$, we obtain

$$G(t) \le x_0(ah+1)^{-\frac{t_0}{h}} + hL \sum_{i=1}^{(t-t_0)/h} (ah+1)^{-\frac{t_0}{h}-i} = \left(x_0 + \frac{L}{a}\right)(ah+1)^{-\frac{t_0}{h}} - \frac{L}{a}(ah+1)^{-\frac{t}{h}} < 0$$

for all $t \ge t_0 + h$, and thus,

$$G(t) < \lim_{t\to\infty} G(t) \le 0$$

for all $t \in h\mathbb{Z}$.

Next we prove the case $-1/h < a < 0$ and $x_0 \geq -L/a$. We have only to prove that $G(t) > 0$ for all $t \leq t_0 - h$. Using $-1/h < a < 0$ and $x_0 \geq -L/a$, we obtain

$$G(t) \geq x_0(ah+1)^{-\frac{t_0}{h}} - hL \sum_{i=1}^{(t_0-t)/h} (ah+1)^{-\frac{t_0}{h}+i-1} = \left(x_0 + \frac{L}{a}\right)(ah+1)^{-\frac{t_0}{h}} - \frac{L}{a}(ah+1)^{-\frac{t}{h}} > 0$$

for all $t \leq t_0 - h$, and thus,

$$G(t) > \lim_{t\to-\infty} G(t) \geq 0$$

for all $t \in h\mathbb{Z}$. □

Lemma 2.3. *Suppose that $a > -1/h$, and $f(t) > 0$ for all $t \in h\mathbb{Z}$. Let $x(t)$ be any solution of (2.1). Then a real-valued function $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h\xi(t) - a\xi(t)| \leq f(t)$ for all $t \in \mathbb{T}^\kappa$ if and only if*

$$0 \leq \Delta_h \left\{(\xi(t) + x(t))(ah+1)^{-\frac{t}{h}}\right\} \leq 2f(t)(ah+1)^{-\frac{t+h}{h}} = 2\Delta_h x(t)(ah+1)^{-\frac{t}{h}}$$

for all $t \in \mathbb{T}^\kappa$.

Proof. The inequalities in Lemma 2.3 is true since $ah + 1 > 0$ and the equality

$$\begin{aligned}\Delta_h \left\{(\xi(t) + x(t))(ah+1)^{-\frac{t}{h}}\right\} &= \frac{1}{h}\left\{(\xi(t+h) + x(t+h)) - (ah+1)(\xi(t) + x(t))\right\}(ah+1)^{-\frac{t+h}{h}} \\ &= (\Delta_h\xi(t) - a\xi(t) + \Delta_h x(t) - ax(t))(ah+1)^{-\frac{t+h}{h}} \\ &= (\Delta_h\xi(t) - a\xi(t) + f(t))(ah+1)^{-\frac{t+h}{h}}\end{aligned}$$

holds for all $t \in \mathbb{T}^\kappa$. If $\xi(t) \equiv 0$ then we have $\Delta_h x(t)(ah+1)^{-\frac{t}{h}} = f(t)(ah+1)^{-\frac{t+h}{h}}$ from the equality above, and thus, this completes the proof. □

Proposition 2.4. *Suppose that $a \neq 0$ and $a > -1/h$, and there exists an $L > 0$ such that $0 < f(t) \leq L$ for all $t \in h\mathbb{Z}$. Suppose also that a function $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h\xi(t) - a\xi(t)| \leq f(t)$ for all $t \in \mathbb{T}^\kappa$. Let $G(t) := F(t; 0, -L/a, f)$, where F is the function given by (2.2). Then*

$$G_0 = \begin{cases} \lim_{t\to\infty} G(t), & \text{if} \quad a > 0, \\ \lim_{t\to-\infty} G(t), & \text{if} \quad -\dfrac{1}{h} < a < 0 \end{cases}$$

exists, and there exist a nondecreasing function $u : \mathbb{T} \to \mathbb{R}$ and a nonincreasing function $v : \mathbb{T} \to \mathbb{R}$ such that

$$\xi(t) = (u(t) + G_0 - G(t))(ah+1)^{\frac{t}{h}} = (v(t) - G_0 + G(t))(ah+1)^{\frac{t}{h}} \tag{2.3}$$

for all $t \in \mathbb{T}$, and the following hold:

- *(i) if $a > 0$ and $\bar{t} := \max\mathbb{T}$ exists, then the inequality $u(t) \leq u(\bar{t}) < v(\bar{t}) \leq v(t)$ holds for all $t \in \mathbb{T}$;*
- *(ii) if $a > 0$ and $\max\mathbb{T}$ does not exist, then $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} v(t)$ exist, and $u(t) \leq \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) \leq v(t)$ holds for all $t \in \mathbb{T}$;*
- *(iii) if $-1/h < a < 0$ and $\underline{t} := \min\mathbb{T}$ exists, then the inequality $v(t) \leq v(\underline{t}) < u(\underline{t}) \leq u(t)$ holds for all $t \in \mathbb{T}$;*
- *(iv) if $-1/h < a < 0$ and $\min\mathbb{T}$ does not exist, then $\lim_{t\to-\infty} u(t)$ and $\lim_{t\to-\infty} v(t)$ exist, and $v(t) \leq \lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} u(t) \leq u(t)$ holds for all $t \in \mathbb{T}$.*

Proof. By means of Lemma 2.2, we see that $G(t) := F(t; 0, -L/a, f)$ is an increasing function on $h\mathbb{Z}$; and $\lim_{t\to\infty} G(t) \le 0$ exists if $a > 0$; $\lim_{t\to-\infty} G(t) \ge 0$ exists if $-1/h < a < 0$; and

$$G(t) - G_0 \begin{cases} < 0, & \text{if} \quad a > 0, \\ > 0, & \text{if} \quad -\dfrac{1}{h} < a < 0 \end{cases} \tag{2.4}$$

for all $t \in h\mathbb{Z}$. Let

$$x(t) = G(t)(ah+1)^{\frac{t}{h}}$$

for all $t \in h\mathbb{Z}$. Then $x(t)$ is the solution of the initial-value problem (2.1) with $x(0) = -L/a$, by Lemma 2.1.

Now we consider the functions

$$u(t) = (\xi(t) + x(t))(ah+1)^{-\frac{t}{h}} - G_0 \quad \text{and} \quad v(t) = (\xi(t) - x(t))(ah+1)^{-\frac{t}{h}} + G_0$$

on $\mathbb{T}$, where $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h \xi(t) - a\xi(t)| \le f(t)$ for all $t \in \mathbb{T}^\kappa$. Then it is clear that (2.3) holds on $\mathbb{T}$. By (2.3), we have

$$u(t) - v(t) = 2x(t)(ah+1)^{-\frac{t}{h}} - 2G_0 = 2(G(t) - G_0) \tag{2.5}$$

for all $t \in \mathbb{T}$. From this and (2.4), we obtain

$$u(t) \begin{cases} < v(t), & \text{if} \quad a > 0, \\ > v(t), & \text{if} \quad -\dfrac{1}{h} < a < 0 \end{cases} \tag{2.6}$$

for all $t \in \mathbb{T}$. Note that $u : \mathbb{T} \to \mathbb{R}$ is a nondecreasing function and $v : \mathbb{T} \to \mathbb{R}$ is a nonincreasing function on $\mathbb{T}$ by using Lemma 2.3.

First, we will prove (i). From the facts above with $\bar{t} = \max \mathbb{T}$, $u(\bar{t})$ and $v(\bar{t})$ become the maximum of $u(t)$ and the minimum of $v(t)$ on $\mathbb{T}$, respectively. Hence, together with (2.6), we see that the claim of (i) is true. The argument for (iii) is similar to that given above for (i). Therefore we omit the proof of (iii).

Next, we prove (ii). Fix the constant $s \in \mathbb{T}$. Since $v : \mathbb{T} \to \mathbb{R}$ is a nonincreasing function on $\mathbb{T}$ and (2.6) holds on $\mathbb{T}$, we have

$$u(t) < v(t) \le v(s) < \infty$$

for $t \ge s$ and $t \in \mathbb{T}$. That is, $u(t)$ is bounded above for $t \ge s$. Since $u : \mathbb{T} \to \mathbb{R}$ is a nondecreasing function, we see that $\lim_{t\to\infty} u(t)$ exists. By (2.5), we obtain the inequality in the claim of (ii). The argument for (iv) is similar to that given above for (ii). Therefore we omit the proof of (iv). The proof is now complete. □

3 Main result

We will present the main theorem and its proof.

Theorem 3.1. *Suppose that $a \ne 0$ and $a > -1/h$, and there exists an $\bar{\varepsilon} > 0$ such that $0 < \phi(t) \le \bar{\varepsilon}$ for all $t \in h\mathbb{Z}$. Suppose also that a function $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h \xi(t) - a\xi(t)| \le \phi(t)$ for all $t \in \mathbb{T}^\kappa$. Let $G(t) := F(t; 0, -\bar{\varepsilon}/a, \phi)$, where F is the function given by (2.2). Then*

$$G_0 = \begin{cases} \lim\limits_{t\to\infty} G(t), & \text{if} \quad a > 0, \\ \lim\limits_{t\to-\infty} G(t), & \text{if} \quad -\dfrac{1}{h} < a < 0 \end{cases}$$

exists, and the following hold:

(i) if $a > 0$ *and* $\bar{t} := \max \mathbb{T}$ *exists, then any solution* $x(t)$ *of* (1.1) *with* $|\xi(\bar{t}) - x(\bar{t})| < (G_0 - G(\bar{t}))(ah + 1)^{\bar{t}/h}$ *satisfies that* $|\xi(t) - x(t)| < (G_0 - G(t))(ah + 1)^{t/h}$ *for all* $t \in \mathbb{T}$*;*

(ii) if $a > 0$ *and* $\max \mathbb{T}$ *does not exist, then* $\lim_{t\to\infty} \xi(t)(ah + 1)^{-t/h}$ *exists, and there exists a unique solution*

$$x(t) = \left\{ \lim_{t\to\infty} \xi(t)(ah + 1)^{-\frac{t}{h}} \right\} (ah + 1)^{\frac{t}{h}}$$

of (1.1) *such that* $|\xi(t) - x(t)| \le (G_0 - G(t))(ah + 1)^{t/h}$ *for all* $t \in \mathbb{T}$*;*

(iii) if $-1/h < a < 0$ *and* $\underline{t} := \min \mathbb{T}$ *exists, then any solution* $x(t)$ *of* (1.1) *with* $|\xi(\underline{t}) - x(\underline{t})| < (G(\underline{t}) - G_0)(ah + 1)^{\underline{t}/h}$ *satisfies that* $|\xi(t) - x(t)| < (G(t) - G_0)(ah + 1)^{t/h}$ *for all* $t \in \mathbb{T}$*;*

(iv) if $-1/h < a < 0$ *and* $\min \mathbb{T}$ *does not exist, then* $\lim_{t\to-\infty} \xi(t)(ah + 1)^{-t/h}$ *exists, and there exists a unique solution*

$$x(t) = \left\{ \lim_{t\to-\infty} \xi(t)(ah + 1)^{-\frac{t}{h}} \right\} (ah + 1)^{\frac{t}{h}}$$

of (1.1) *such that* $|\xi(t) - x(t)| \le (G(t) - G_0)(ah + 1)^{t/h}$ *for all* $t \in \mathbb{T}$.

Proof. From Lemma 2.2, we see that G_0 exists and

$$G(t) - G_0 = F\left(t; 0, -\frac{\overline{\varepsilon}}{a}, \phi\right) - G_0 \begin{cases} < 0, & \text{if} \quad a > 0, \\ > 0, & \text{if} \quad -\dfrac{1}{h} < a < 0 \end{cases} \tag{3.1}$$

for all $t \in h\mathbb{Z}$. By Proposition 2.4, we can find a nondecreasing function $u : \mathbb{T} \to \mathbb{R}$ and a nonincreasing function $v : \mathbb{T} \to \mathbb{R}$ such that

$$\xi(t) = (u(t) + G_0 - G(t))(ah + 1)^{\frac{t}{h}} = (v(t) - G_0 + G(t))(ah + 1)^{\frac{t}{h}} \tag{3.2}$$

for all $t \in \mathbb{T}$.

First we prove case (i). We consider any solution $x_1(t)$ of (1.1) with

$$|\xi(\bar{t}) - x_1(\bar{t})| < (G_0 - G(\bar{t}))(ah + 1)^{\frac{\bar{t}}{h}}, \tag{3.3}$$

where $a > 0$ and $\bar{t} := \max \mathbb{T}$. Note here that $G_0 - G(\bar{t})$ is positive by (3.1), and $x_1(t)$ is written as

$$x_1(t) = x_1(\bar{t})(ah + 1)^{\frac{t-\bar{t}}{h}}$$

for $t \in \mathbb{T}$, so that this together with Proposition 2.4 (i), (3.2) and (3.3) implies that

$$u(t) \le u(\bar{t}) < x_1(\bar{t})(ah + 1)^{-\frac{\bar{t}}{h}} < v(\bar{t}) \le v(t)$$

for $t \in \mathbb{T}$. From this and (3.2), we obtain

$$(\xi(t) - x_1(t))(ah + 1)^{-\frac{t}{h}} = u(t) + G_0 - G(t) - x_1(\bar{t})(ah + 1)^{-\frac{\bar{t}}{h}} < G_0 - G(t)$$

and

$$(\xi(t) - x_1(t))(ah + 1)^{-\frac{t}{h}} = v(t) - G_0 + G(t) - x_1(\bar{t})(ah + 1)^{-\frac{\bar{t}}{h}} > -(G_0 - G(t))$$

for $t \in \mathbb{T}$. Thus, the claim of (i) is true.

Next we will prove case (ii). Using Proposition 2.4 (ii), we see that $\lim_{t\to\infty} u(t)$ and $\lim_{t\to\infty} v(t)$ exist, and

$$u(t) \le \lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) \le v(t)$$

holds for all $t \in \mathbb{T}$. From this and (3.2), we conclude that

$$\lim_{t\to\infty} \xi(t)(ah + 1)^{-\frac{t}{h}} = \lim_{t\to\infty} (u(t) + G_0 - G(t)) = \lim_{t\to\infty} u(t)$$

exists and

$$u(t) \le \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \le v(t)$$

for all $t \in \mathbb{T}$. Now we consider the solution

$$x_2(t) = \left\{ \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \right\} (ah+1)^{\frac{t}{h}}$$

of (1.1) on $h\mathbb{Z}$. Using (3.2) and the inequality above, we have

$$(\xi(t) - x_2(t))(ah+1)^{-\frac{t}{h}} = u(t) + G_0 - G(t) - \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \le G_0 - G(t)$$

and

$$(\xi(t) - x_2(t))(ah+1)^{-\frac{t}{h}} = v(t) - G_0 + G(t) - \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \ge -(G_0 - G(t))$$

for $t \in \mathbb{T}$, and thus, $|\xi(t) - x_2(t)| \le (G_0 - G(t))(ah+1)^{\frac{t}{h}}$ for all $t \in \mathbb{T}$.

We next prove the uniqueness of $x_2(t)$. By way of contradiction, we consider a solution $y(t)$ of (1.1) such that $y(t) \not\equiv x_2(t)$ and $|\xi(t) - y(t)| \le (G_0 - G(t))(ah+1)^{\frac{t}{h}}$ for all $t \in \mathbb{T}$. Since the uniqueness of solutions of (1.1) are guaranteed for the initial value problem, we can rewrite $y(t)$ as

$$y(t) = c(ah+1)^{\frac{t}{h}}$$

for $t \in h\mathbb{Z}$, where $c \ne \lim_{t\to\infty} \xi(t)(ah+1)^{-t/h}$. Therefore, we obtain

$$\begin{aligned} 0 \ne \left| c - \lim_{t\to\infty} \xi(t)(ah+1)^{-\frac{t}{h}} \right| &= |y(t) - x_2(t)|(ah+1)^{-\frac{t}{h}} \le (|y(t) - \xi(t)| + |\xi(t) - x_2(t)|)(ah+1)^{-\frac{t}{h}} \\ &\le 2(G_0 - G(t)) \end{aligned}$$

for all $t \in \mathbb{T}$, however, this contradicts the fact that $\lim_{t\to\infty} G(t) = G_0$. Thus, the claim of (ii) is true.

Using the same arguments in (i) and (ii), we can conclude that the claims of (iii) and (iv) are also true. The proof of Theorem 3.1 is now complete. □

Theorem 3.1 implies the following result.

Corollary 3.2. *Suppose that $a \ne 0$ and $a > -1/h$, and there exists an $\bar{\varepsilon} > 0$ such that $0 < \phi(t) \le \bar{\varepsilon}$ for all $t \in h\mathbb{Z}$. Suppose also that a function $\xi : \mathbb{T} \to \mathbb{R}$ satisfies $|\Delta_h \xi(t) - a\xi(t)| \le \phi(t)$ for all $t \in \mathbb{T}^\kappa$. Then there exist a positive function $\psi : \mathbb{T} \to \mathbb{R}$ and a solution $x : \mathbb{T} \to \mathbb{R}$ of* (1.1) *such that $|\xi(t) - x(t)| \le \psi(t)$ for all $t \in \mathbb{T}$.*

Remark 3.1. The statement of Corollary 3.2 do not exactly match HURS because of the restriction of the function $\phi(t)$. Therefore, the future subject is to consider whether the boundedness of $\phi(t)$ is necessary. However, it can be seen that the above Theorem 3.1 and Corollary 3.2 include Theorem A and Corollary B, respectively.

Now we will show that this claim. Consider the case where $\phi(t) \equiv \varepsilon$ and $x_0 = \varepsilon/a$. Since

$$G(t) = F\left(t; 0, -\frac{\varepsilon}{a}, \varepsilon\right) = -\frac{\varepsilon}{a} + h\varepsilon \sum_{i=1}^{t/h} (ah+1)^{-i} = -\frac{\varepsilon}{a}(ah+1)^{-\frac{t}{h}}$$

holds if $t \ge t_0 + h$, and

$$G(t) = F\left(t; 0, -\frac{\varepsilon}{a}, \varepsilon\right) = -\frac{\varepsilon}{a} - h\varepsilon \sum_{i=1}^{-t/h} (ah+1)^{i-1} = -\frac{\varepsilon}{a}(ah+1)^{-\frac{t}{h}}$$

holds if $t \le t_0 - h$, we see that

$$G(t) = -\frac{\varepsilon}{a}(ah+1)^{-\frac{t}{h}}$$

for all $t \in h\mathbb{Z}$. Then we have $G_0 = 0$, so that

$$|G_0 - G(t)|(ah+1)^{\frac{t}{h}} = \frac{\varepsilon}{|a|}$$

for all $t \in h\mathbb{Z}$. Therefore, we can conclude that Theorem 3.1 includes Theorem A. Obviously, Corollary B implies Corollary 3.2.

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