On some behaviors of irrational rotations

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Contents

1. **Introduction**

The investigations for uniformly distributed sequences have a long history of more than one hundred years. The Kronecker's approximation theorem is well-known as the starting point of the development result of the theory of uniformly distributed sequences. In 1884, Kronecker proved that irrational rotations are dense in the unit interval. At the beginning of 1900's refinements and generalizations of Kronecker's theorem were established by various researchers such as Bohl, Sierpinski, Hardy–Littlewood, Weyl etc. Especially, in 1916, Weyl [27] showed that irrational rotations are uniformly distributed sequences mod 1, who used the method of Fourier analysis. After his results, the investigations for uniformly distributed sequences, especially irrational rotations, were developed in the number theory, the probability theory and various fields. For example, many researchers, such as Hardy–Littlewood, Ostrowski, Hecke, Behnke etc, gave their results for the lattice point problems, the problems of Diophantine approximation and so on. Particularly, Ostrowski introduced an expansion of a natural number based on continued fraction expansion for an irrational number, later called as Ostrowski expansion. Mori and Takashima [15] recently studied the distributions of leading digits of nth power of some integers, with applying the approximation of an irrational number α by its nth convergent.

In this dissertation, we develop their idea of [15] and we introduce its refinement, "rational rotation approximation". We consider not only the rational rotation approximation but also the Ostrowski expansion and we study important behaviors of irrational rotations, such as the position of point of the irrational rotation itself, the behaviors of partial sums of irrational rotations and the behaviors of discrepancies of irrational rotations. In case that an irrational number α has large partial quotients, our methods are very useful for studies of irrational rotations and we give some formulas and estimates for the above behaviors. Furthermore, we give the mathematical explanations for the unusual behaviors of irrational rotations based on such α.

In Section 3, we apply the continued fraction expansion and rational rotation approximation to observe the behavior of the position of each point of irrational rotations. It is easily seen that rational rotations $\left\{i\frac{p_n}{q_n}\right\}$ $(i = 1, 2, 3, \dots)$ are periodic, where $\frac{p_n}{q_n}$ is the nth convergent of an irrational number and $\{x\}$ denotes fractional part of a real number x. By applying the periodicity of $\left\{i\frac{p_n}{q_n}\right\}$, we give the exact formula which determines the position of each point of an irrational rotation.

Note that the mean of the uniform distribution on $[0, 1)$ is equal to $\frac{1}{2}$ and the variance of the uniform distribution is equal to $\frac{1}{12}$. In Section 4, we discuss the first order partial sums $\sum_{i=1}^{n}\left(\{i\alpha\} - \frac{1}{2}\right)$: the fluctuation from the mean $\frac{1}{2}$, and we also discuss the second order partial sums $\sum_{i=1}^{n}\left\{\left(\{i\alpha\} - \frac{1}{2}\right)^2 - \frac{1}{12}\right\}$: the fluctuation from the variance $\frac{1}{12}$. The studies of the partial sums $\sum_{i=1}^{n}\left(\{i\alpha\} - \frac{1}{2}\right)$ were initiated by Hardy–Littlewood [5], Ostrowski [18], Hecke [7], Behnke [2], Khintchine [10], [11] and so on in the early 20th century. We give an exact formula for the above sum $\sum_{i=1}^{n}\left(\{i\alpha\} - \frac{1}{2}\right)$ by applying rational rotation approximation, decomposition based on Ostrowski expansion and cancellation techniques. Our calculations are different from those in Ostrowski [18] because we use cancellation techniques while Ostrowski used direct calculations. Especially, for irrational rotations based on α with large partial quotients, we give mathematical explanations for "quadratic-function-like" repetitions shown in Fig.1 and Fig.2.

It is well-known that $\pi - 3$ has $[0; 7, 15, 1, 292, 1, 1, \dots]$ as its continued fraction expansion. In case that $\alpha = \pi - 3$, the behaviors of the partial sums show positive "quadratic-function-like" repetitions caused by its 4th large partial quotient 292. We also know that $\log_{10} 7$ has $[0; 1, 5, 2, 5, 6, 1, 4813, 1, 1, \dots]$ as its continued fraction expansion . In case that $\alpha = \log_{10} 7$, the behaviors of the partial sums show negative "quadratic-function-like" repetitions caused by its 7th large partial quotient 4813.

Next, we consider the second order partial sums $\sum_{i=1}^{n}\left\{\left(\{i\alpha\} - \frac{1}{2}\right)^2 - \frac{1}{12}\right\}$. We give an exact formula for such sums, slightly different from the formula in [21]. Moreover, we provide some estimates from our exact formula so that we explain the effects of large partial quotients on the strange "cubic-function-like" repetitions of the sums shown in Fig.3, Fig.4, Fig.5 and Fig.6.

In Section 5, we consider the estimates of discrepancies of irrational rotations. The notion of discrepancy was introduced to measure the speed of convergence to the uniform distribution.

Setokuchi and Takashima [23] and Setokuchi [22] gave estimates for discrepancies of irrational rotations with a single isolated large partial quotient by refining Schoissengeier's results ([19], [20]). They gave some mathematical explanations of appearance of quadratic curves and they call such curves *hills*. Furthermore, Setokuchi [22] studied that such hills appear many times in long term behavior of discrepancies under specific conditions.

On the other hand, we give the estimates for discrepancies of irrational rotations by rational rotation approximation. First, we use the method of rational rotation approximation and we give a rough upper bound for the discrepancy of irrational rotations in terms of the continued fraction expansion of α and the related Ostrowski expansion. Then, we give simply an another proof of Weyl's lemma.

Secondly, we consider the estimate of discrepancies of irrational rotations with single isolated large partial quotient by rational rotation approximation. Instead of using Schoissengeier's results, we use the simple application of rational rotation approximation. We give much more accurate estimates for discrepancies of irrational rotations. We show that the first part of the graph of discrepancies of irrational rotations with a single isolated large partial quotient is linearly decreasing, provided we observe the discrepancies on a linear scale with suitable step. We also prove that large hills, caused by single isolated large partial quotients, will appear infinitely often.

Next, we discuss the estimates of discrepancies of irrational rotations with several large partial quotients, caused by several, but not single, larger partial quotients. [22], [23] considered some irrational numbers which have a single isolated large partial quotient, for example, $\log_{10} 7$ (or $1 - \log_{10} 7$) and $\pi - 3$. Let us see their continued fraction expansions in more detail.

$$\begin{aligned}\log_{10} 7 = [0; &1, 5, 2, 5, 6, 1, 4813, 1, 1, 2, 2, 2, 1, 1, 1, 6, 5, 1, 83, 7, 2, 1, 1, 1, 8, 5,\\ &21, 1, 1, 3, 2, 1, 4, 2, 3, 14, 2, 6, 1, 1, 5, 2, 1, 2, 4, 26, 2, 6, 1, 5, 1, 1,\\ &2, 2, 3, 6, 2, 2, 103, 2, 2, 1084, 1, 1, 1, 1, 12, 1, 8, 5, 1, 3, 4, \ldots],\end{aligned}$$

$$\pi - 3 = [0; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1,$$
$$15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, \ldots].$$

$\log_{10} 7$ has large partial quotients not only $a_7 = 4813$ but also comparatively large partial quotients such as $a_{19} = 83, a_{59} = 103$ and $a_{62} = 1084$. Similarly, $\pi - 3$ has several large partial quotients, such as $a_4 = 292, a_{21} = 84$ and $a_{33} = 99$. Furthermore, we have another example, $\log_{10} 37 - 1$, whose continued fraction expansion is given as follows:

$$\log_{10} 37 - 1 = [0; 1, 1, 3, 6, 25, 1, 3, 1, 2, 1, 248, 140, 1, 85, 1, 4, 2, 4, 1, 8, 4, 1, \ldots].$$

Note that $\log_{10} 37 - 1$ has two continued large partial quotients such as $a_{11} = 248, a_{12} = 140$ and one large partial quotient $a_{14} = 85$. We want to study effects on behaviors of discrepancies, caused by two such continued large partial quotients and one large partial quotient. [22], [23] and [24] contain mainly the cases where α has one single isolated large partial quotient. It would seem to be difficult to discuss the cases where α has several large partial quotients by using Schoissengeier's results (cf. [22], [23]). We consider the cases where α has several large partial quotients and give some estimates for discrepancies, by using rational rotation approximation and the Ostrowski expansion. In such cases, the way of overlapping of hills depends on the orders of several large partial quotients, even or odd. We give some graphs of compound quadratic curves of behavior of discrepancies and we explain effects of such several large partial quotients mathematically, by using our estimates.

The following sections of this dissertation are available as articles with minor modifications.

Section 3:
Shimaru, N. and Takashima, K.: Continued fractions and irrational rotations, *Periodica Math. Hungr.*, **75 (2)**, (2017), 155 – 158.

Section 4:
Mori, Y., Shimaru, N. and Takashima, K.: On the distribution of partial sums of

irrational rotations, *Periodica Math.Hungr.*, **in print**.

Section 5:

Doi, K., Shimaru, N. and Takashima, K., On upper estimates for discrepancies of irrational rotations: via Rational Rotation Approximations, *Acta Math Hungr.*, **152 (1)**, (2017), 109 – 113.

Shimaru, N. and Takashima, K.: Continued fractions and irrational rotations, *Periodica Math. Hungr.*, **75 (2)**, (2017), 155 – 158.

Shimaru, N. and Takashima, K.: On discrepancies of irrational rotations with several large partial quotients, *Acta Math. Hungr.*, **156 (2)**, (2018), 449 – 458.

2. **Preliminaries**

In this section, we prepare well-known notions, notations and results for uniformly distributed sequences and continued fractions, as quickly references.

2.1. **Uniformly distributed sequences**

We recall definition of uniformly distributed sequences, Weyl's criterion (Weyl's lemma) and basic properties of discrepancies (cf. [14]).

For a real number x, the fractional part $\{x\}$ of x is defined by $\{x\} = x - [x]$, where $[x]$ denotes the integral part of x.

Definition 2.1. A given sequence (x_n), $n = 1, 2, 3, \ldots$, of real numbers is said to be uniformly distributed mod 1 (abbreviated u.d. mod 1) if for the every pair a, b of real numbers with $0 \leq a < b \leq 1$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{I}_{[a,b)}(\{x_n\}) = b - a, \tag{2.1}$$

where $\mathbf{I}_{[a,b)}$ is the indicator function of the interval $[a, b)$.

Theorem 2.2 (cf. [27], Weyl's criterion)**.** *A sequence (x_n), $n = 1, 2, \ldots$ is u.d. mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

holds for all integer $h \neq 0$.

Especially, for any irrational number α, Weyl applied this theorem to $\{n\alpha\}$, $n = 1, 2, \ldots$. Then, Weyl's criterion implies the next lemma.

Lemma 2.3 (cf. [27], Weyl's lemma)**.** *Let α be an irrational number. Then, irrational rotation $\{n\alpha\}$ is u.d. mod 1.*

For a sequence (x_n), the discrepancies have usually two types of definition.

Definition 2.4 (cf. [14]).

$$D_N(\{x_n\}) = \sup_{0\le a<b<1} \left| \frac{1}{N} \sum_{n=1}^{N} \mathbf{I}_{[a,b)}(\{x_n\}) - (b-a) \right|,$$

$$D_N^*(\{x_n\}) = \sup_{0\le a<1} \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{I}_{[0,a)}(\{x_n\}) - a \right|.$$

The essential point of concept of discrepancy is that the notion of uniformly distributed sequences can be covered by it; i.e. the convergence in (2.1) is uniform with respect to all intervals $[a,b) \subseteq [0,1)$.

$$\lim_{N\to\infty} D_N(\{n\alpha\}) = \lim_{N\to\infty} D_N^*(\{n\alpha\}) = 0. \tag{2.2}$$

In Section 5, we discuss mainly $D_N^*(\{n\alpha\})$ and $ND_N^*(\{n\alpha\})$.

2.2. Continued fractions

For an irrational number α $(0<\alpha<1)$, an expression of a continued fraction is denoted by as follows:

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cdots}}}}},$$

or simply,

$$\alpha = [\ 0\ ;\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots\ ].$$

We denote the nth partial quotient by a_n and the nth convergent of α by $r_n = p_n/q_n = [\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a_n\ ]$.

Now, we define continued fraction transformation $\tau : [0,1) \longrightarrow [0,1)$.

Definition 2.5 (cf. [8]).

$$\tau(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $[x]$ is integral part of x.

We have the following relations by using inductively the continued fraction transformation:

$$\tau^{n-1}(\alpha) = [0\ ;\ a_n,\ a_{n+1},\ a_{n+2},\ \ldots]\ ,\ \frac{1}{\tau^{n-1}(\alpha)} = a_n + \tau^n(\alpha),$$

where τ^0 is identity map and τ^n is composed n times with itself. We imply that continued fraction transformation τ is algorithm to derive the partial quotient a_n.

The numerator p_n and the denominator q_n of the nth convergent have the following recursion formulas:

Theorem 2.6 (cf. [8]). *Let n be a natural number.*

$$\begin{cases} p_{n+1} = a_{n+1}p_n + p_{n-1} \quad , \quad p_0 = 0\ ,\ p_1 = 1, \\ q_{n+1} = a_{n+1}q_n + q_{n-1} \quad , \quad q_0 = 1\ ,\ q_1 = a_1. \end{cases} \tag{2.3}$$

Proof. Let us assume that $p_k/q_k = [\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a_k\ ]$ is true for $^\forall k \leq n$.

$$[\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a_{k+1}\ ] = [\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a_k + \tfrac{1}{a_{k+1}}\ ].$$

Now, put $a_k + 1/a_{k+1} = a'_k$,

$$\begin{aligned} [\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a_{k+1}\ ] &= [\ 0\ ;\ a_1,\ a_2,\ \ldots\ ,\ a'_k\ ] \\ &= \frac{a'_k p_{k-1} + p_{k-2}}{a'_k q_{k-1} + q_{k-2}} \\ &= \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} \\ &= \frac{p_{k+1}}{q_{k+1}}. \end{aligned}$$

□

Now, note that by above properties, we also have

$$\begin{aligned} \alpha &= [\ 0\ ;\ a_1\ ,\ a_2\ ,\ \ldots\ ,\ a_n\ +\ \tau^n(\alpha)\ ] \\ &= \frac{(a_n\ +\ \tau^n(\alpha))p_{n-1} + p_{n-2}}{(a_n\ +\ \tau^n(\alpha))q_{n-1} + q_{n-2}} \\ &= \frac{p_n + \tau^n(\alpha)p_{n-1}}{q_n + \tau^n(\alpha)q_{n-1}}. \end{aligned} \tag{2.4}$$

From (2.3), we obtain the next well–known relation.

Theorem 2.7 (cf. [8])**.** *For any positive integer n,*

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, \tag{2.5}$$

and

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}.$$

Proof. It is trivial from (2.3). □

We obtain immediately the following formulas from (2.3) and (2.5)

Corollary 2.8 (cf. [13])**.** *For any positive integer n,*

$$p_n q_{n-2} - q_n p_{n-2} = (-1)^{n-2} a_n, \tag{2.6}$$

and

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-2} a_n}{q_n q_{n-2}}.$$

Remark 2.1. From (2.5), since right–hand side is positive for odd n, it is easily seen that every odd–order convergent is greater than any even–order convergent. From (2.6), we can easily see that even–order convergents are increasing and odd–order convergents are decreasing i.e.

$$r_2 < r_4 < \cdots < r_{2n} < \cdots < \alpha < \cdots < r_{2n+1} < \cdots < r_3 < r_1.$$

Furthermore, on the basis of (2.5) and (2.6), we can easily derive the estimate between α and p_n/q_n.

Theorem 2.9 (cf. [8])**.** *For any positive integer n, let α be an irrational number. We have the inequality;*

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Corollary 2.10 (cf. [8])**.** *For any positive integer n,*

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n \tau^n(\alpha)}{q_n (q_n + \tau^n(\alpha) q_{n-1})}.$$

Proof. Let recall that $1/\tau^n(\alpha) = a_{n+1} + \tau^{n+1}(\alpha)$ and (2.4),

$$\begin{aligned}\alpha - \frac{p_n}{q_n} &= \frac{(-1)^n \tau^n(\alpha)}{q_n(q_n + \tau^n(\alpha) q_{n-1})} \\ &= \frac{(-1)^n}{q_n(q_n(a_{n+1} + \tau^{n+1}(\alpha)) + q_{n-1})} \\ &= \frac{(-1)^n}{q_n(q_{n+1} + q_n \tau^{n+1}(\alpha))}.\end{aligned}$$

□

By continued fraction transformation, it is easily seen that $0 < \tau^n(\alpha) < 1$. Hence, we have the proposition.

Proposition 2.11 (cf. [8]). *For any positive integer n,*

$$\frac{1}{q_n(q_{n+1} + q_n)} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}.$$

Now, we introduce the expansion for natural number N that is an important idea for investigation of irrational rotations (cf. [17], [18]). Let the natural number m be chosen by the inequality, $q_m \leq N < q_{m+1}$, and N has the expansion as follows:

$$N = \sum_{j=0}^{m} b_j q_j \quad , \quad q_0 = 1, \tag{2.7}$$

where each coefficient b_j satisfies

$$0 \leq b_0 < a_1,\ 0 \leq b_j \leq a_{j+1},\ j \geq 1,\ \text{and } b_{j-1} = 0 \text{ if } b_j = a_{j+1} \quad .$$

This expansion is called as Ostrowski expansion with respect to continued fraction expansion of α. In all that follows, m and j are the number introduced in Ostrowski expansion.

Mori and Takashima [15] used the idea of rational rotation approximation to investigate behaviors of irrational rotations. Furthermore, we extend and clarify the method of rational rotation approximations for irrational rotations. We, now, summarize rational rotation approximation method as follows:

Lemma 2.12 (cf. [3]). *Let α be an irrational number. For each $i = 1, 2, \cdots, q_n$, let k_i denote an integer satisfying $ip_n \equiv k_i \pmod{q_n}$, $0 \le k_i < q_n$. We have*

$$\frac{k_i}{q_n} < \{i\alpha\} < \{(i+q_n)\alpha\} < \{(i+2q_n)\alpha\} < \cdots < \frac{k_i+1}{q_n},$$

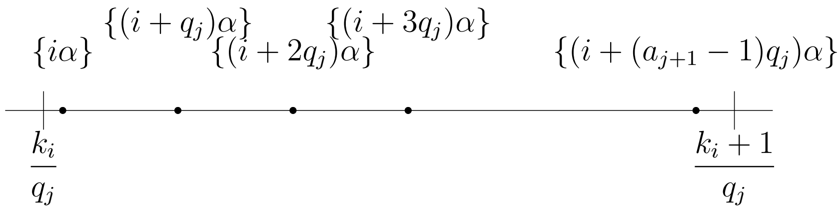
in case n is even, and

$$\frac{k_i}{q_n} < \cdots < \{(i+2q_n)\alpha\} < \{(i+q_n)\alpha\} < \{i\alpha\} < \frac{k_i+1}{q_n},$$

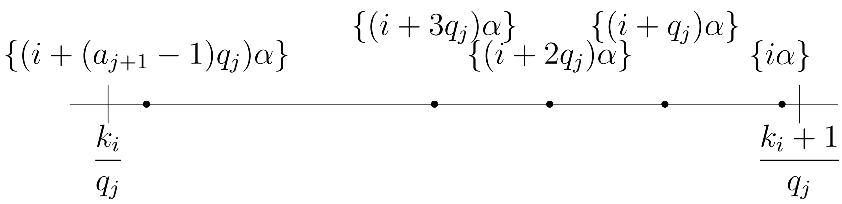
in case n is odd.

In this lemma, we want to emphasize that the distance between neighboring two points is equal to Δ_n, where $\delta_n = |\alpha - p_n/q_n|$ and $\Delta_n = \{q_n\alpha\} = q_n\delta_n$.

j : even



j : odd



3. Continued fractions and irrational rotations

We discuss the behaviors of irrational rotations $\{N\alpha\}$ by using method that is approximated by the nth convergent of α. We provide the first formula in Theorem3.3 for $p_nq_k - q_np_k \ \ (k < n)$ in terms of polynomials Q_i defined in terms of a_j's. Furthermore, We use the formula for $p_nq_k - q_np_k$ and Ostrowski expansion to derive the exact formula for $\{N\alpha\}$ (Theorem3.5).

We define polynomials Q_i based on partial quotients a_j's for $i < n$ as follows:

Definition 3.1 (cf. [25])**.**

$$Q_i(a_n, \cdots, a_{n-i+1}) = \begin{cases} 0 & \text{if } \ i = -1, \\ 1 & \text{if } \ i = 0, \\ a_n & \text{if } \ i = 1, \\ a_nQ_{i-1}(a_{n-1}, \cdots, a_{n-i+1}) & \\ \qquad +Q_{i-2}(a_{n-2}, \cdots, a_{n-i+1}) & \text{if } \ 2 \le i \le n-1. \end{cases}$$

From Definition 3.1, it is easily seen that $Q_n(a_1, \cdots, a_n) = q_n$ and $Q_{n-1}(a_2, \cdots, a_n) = p_n$, that is, nth convergent is expressed by

$$\frac{p_n}{q_n} = \frac{Q_{n-1}(a_2, \cdots, a_n)}{Q_n(a_1, \cdots, a_n)}.$$

The polynomials Q_n has the following symmetrical property in its arguments:

Proposition 3.2 (cf. [8])**.** *For any* $n \ge 1$*,*

$$Q_n(a_1, \cdots, a_n) = Q_n(a_n, \cdots, a_1).$$

Proof. The proof follows immediately from induction. □

By using the polynomials Q_i, we have the following generalized formulas for $p_nq_k - q_np_k$:

Theorem 3.3 (cf. [25]). *For any positive integer k satisfied $1 \leq k \leq n$,*

$$p_n q_k - q_n p_k = (-1)^k Q_{n-k-1}(a_{n-1}, a_{n-2}, \cdots, a_{k+2}),$$

and

$$q_k \beta = q_k \frac{p_n}{q_n} \equiv \frac{(-1)^k Q_{n-k-1}(a_n, \cdots, a_{k+2})}{q_n} \pmod 1.$$

Proof. Let suppose that $p_n q_\ell - q_n p_\ell = (-1)^\ell Q_{n-\ell-1}(a_{n-1}, a_{n-2}, \cdots, a_{\ell+2})$ is true for $^\forall \ell \leq n$.

$$\begin{aligned} p_n q_k - q_n p_k &= (a_n p_{n-1} + p_{n-2}) q_k - (a_n q_{n-1} + q_{n-2}) p_k \\ &= a_n (p_{n-1} q_k - q_{n-1} p_k) + (p_{n-2} q_k - q_{n-2} p_k) \\ &= a_n (-1)^k Q_{n-k-2}(a_{n-1}, a_{n-2}, \cdots, a_{k+2}) \\ &\qquad + (-1)^k Q_{n-k-3}(a_{n-2}, a_{n-3}, \cdots, a_{k+2}) \\ &= (-1)^k Q_{n-k-1}(a_n, a_{n-1}, \cdots, a_{k+2}) \quad . \end{aligned}$$

□

We combine Theorem 3.3 with Ostrowski expansion. Let β be the nth convergent p_n/q_n of α. Then, we obtain the following theorem on behaviors of rational rotations.

Theorem 3.4 (cf. [25]). *For $^\forall n > m$,*

$$N\beta \equiv \sum_{j=0}^{m} b_j \frac{(-1)^j Q_{n-j-1}(a_n, \cdots, a_{j+2})}{q_n} \pmod 1.$$

Now, we consider the behaviors of irrational rotation $\{N\alpha\}$. By using the trivial relation, $\alpha = (\alpha - \beta) + \beta$, we obtain the formula on behaviors of irrational rotations.

Theorem 3.5 (cf. [25]). *For $^\forall n > m$,*

$$N\alpha \equiv N(\alpha - \beta) + \sum_{j=0}^{m} b_j \frac{(-1)^j Q_{n-j-1}(a_n, \cdots, a_{j+2})}{q_n} \pmod 1.$$

In this theorem we choose N first, which determines m and the Ostrowski expansion. We want to emphasize the fact that this sum of Theorem 3.5 can be definitely determined by using inductive relation of Q_i's. We choose an integer n sufficiently large than integer m. Then each point $\{N\beta\}$ is one of a fraction with the form

ℓ/q_n $(0 \le \ell < q_n)$, where ℓ can be determined by Q_i's. Note that we can select n large enough, so that we can make the error term $N(\alpha - \beta)$ as small as we want. Thus, we can see exactly where $\{N\alpha\}$ is in the interval $[0, 1)$ by using this Theorem 3.5.

Now, we apply Ostrowski expansion directly to the relation

$$q_j\alpha = q_j\alpha - p_j \qquad (\text{mod } 1)$$

yields for $n > m$ another formula for $\{N\alpha\}$:

Theorem 3.6 (cf. [18], [14]). *For* $^\forall n > m$,

$$N\alpha \equiv \sum_{j=0}^{m} b_j(q_j\alpha - p_j) \qquad (\text{mod } 1).$$

The Theorem 3.6 is similar to the Theorem 3.5, but it is not simple to determined values in this summation. Thus, it would not be easy to determine exactly where $\{N\alpha\}$ exists in the interval $[0, 1)$. Theorem 3.5 seems more useful than Theorem 3.6.

4. On the distribution of partial sums

4.1. Fluctuations from $\frac{1}{2}$, $\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right)$

We consider the sum $\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right)$, and give its exact formula. Especially Ostrowski gave the estimate by direct calculation. We use decomposition by Ostrowski expansion and cancellation technique, and we obtain the following exact formula:

Theorem 4.1 (cf. [16]). *Let α be an irrational number, $0<\alpha<1$, and m be the number determined by Ostrowski expansion, for any natural number N.*

$$\begin{aligned}\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right) &= \sum_{j:odd,1\le j<m}\left(b_jq_js_j-b_j\delta_jq_j^*-\frac{q_j}{2}\Delta_jb_j(b_j-1)+\frac{b_j}{2}\right)\\ &+\sum_{j:even,1\le j<m}\left(b_jq_js_j+b_j\delta_jq_j^*+\frac{q_j}{2}\Delta_jb_j(b_j-1)-\frac{b_j}{2}\right)\\ &+(-1)^m\left(b_m\delta_mq_m^*+\frac{q_m}{2}\Delta_mb_m(b_m-1)-\frac{b_m}{2}+\frac{b_0(b_0+1)}{2}\alpha-\frac{b_0}{2}\right)\end{aligned}$$

where $q_j^*=(q_j(q_j+1))/2$.

Proof. We now decompose the sum $\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right)$ by Ostrowski expansion as follows:

$$\begin{aligned}\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right) &= \sum_{j=0}^{m}\sum_{i=n_j+1}^{n_j+b_jq_j}\left(\{i\alpha\}-\frac{1}{2}\right)\\ &= \sum_{j=0}^{m}\sum_{i=1}^{b_jq_j}\left(\{s_j+i\alpha\}-\frac{1}{2}\right)\\ &= \sum_{j=0}^{m}\sum_{i=1}^{b_jq_j}\left(s_j+\{i\alpha\}-\frac{1}{2}\right)\\ &= \sum_{j=0}^{m}\left(b_jq_js_j+\sum_{i=1}^{b_jq_j}\left(\{i\alpha\}-\frac{1}{2}\right)\right),\end{aligned}$$

where $n_j=\sum_{k=j+1}^{m}b_kq_k$, $s_j=\sum_{k=j+1}^{m}b_kq_k(\alpha-r_k)$ and $s_m=0$. Let us recall that $\alpha-r_j>0$, if j is even, and that $\alpha-r_j<0$, if j is odd (cf. [25]). We can easily show that s_j is enough smaller than Δ_j. We have to divide our arguments into two cases.

Case I: We first assume that j is even.

It is easily seen that for each point k/q_j there is one point k'/q_j which takes symmetric position to k/q_j with respect to the mid-point $1/2$. Let us consider the mid-point of a sub-interval, in which a point $\{i\alpha\}$ is included, and consider also the difference between $\{i\alpha\}$ and the mid-point of the sub-interval. Under these considerations, it is easily seen that the sum is calculated as follows:

$$\sum_{i=1}^{b_j q_j}\left(\{i\alpha\}-\frac{1}{2}\right)=\sum_{\ell=0}^{b_j-1}\left\{\sum_{i=1}^{q_j} i\delta_j+q_j\left(\ell\Delta_j-\frac{1}{2q_j}\right)\right\}.$$

Let $q_j^*=(q_j(q_j+1))/2$. It is also clear that

$$\sum_{i=1}^{q_j} i\delta_j=\delta_j\sum_{i=1}^{q_j} i=\delta_j\frac{q_j(q_j+1)}{2}=\delta_j q_j^*.$$

Thus, we have the following estimate:

$$\sum_{i=1}^{b_j q_j}\left(\{i\alpha\}-\frac{1}{2}\right)=b_j\delta_j q_j^*+\frac{q_j}{2}\Delta_j b_j(b_j-1)-\frac{b_j}{2}.$$

The right-hand side of this equation seems to be a quadratic function of b_j, $0\le b_j\le a_{j+1}$, and its values are non-positive.

Case II. Secondly, we assume that j is odd.

In this case, we have to pay attention on the fact that $\alpha-r_j<0$. Let $\delta_j=|\alpha-r_j|$ again. In this case we only have to modify the above arguments, by considering reverse sign.

$$\sum_{i=1}^{b_j q_j}\left(\{i\alpha\}-\frac{1}{2}\right)=-b_j\delta_j q_j^*-\frac{q_j}{2}\Delta_j b_j(b_j-1)+\frac{b_j}{2}.$$

The right-hand side of the above equation seems also a quadratic function of b_j, and this time, its values are non-negative.

Summing up the above arguments, we have our result. □

We provide Corollary 4.2 to give a mathematical explanation for the graphs shown in Fig.1 and Fig.2.

Corollary 4.2 (cf. [16])**.** *For* $0 \le \nu < a_{j+1}$*, we have*

$$\sum_{i=1}^{\nu q_j} \left(\{i\alpha\} - \frac{1}{2} \right) = (-1)^j \frac{a_{j+1}}{2} (1+\varepsilon_1) x_\nu \, (x_\nu - 1) + \varepsilon_2,$$

where $|\varepsilon_1|$, $|\varepsilon_2| < \dfrac{1}{q_j}$, *and* $x_\nu = \nu / a_{j+1}$, $0 < x_\nu < 1$.

From Corollary 4.2, we can easily see that the sums $\sum_{i=1}^{\nu q_j} \left(\{i\alpha\} - \frac{1}{2} \right)$ behave a quadratic curve for x_ν

4.2. Fluctuations from $\frac{1}{12}$, $\sum_{i=1}^{N} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$

We consider the distribution of $\sum_{i=1}^{N} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$, and we give its exact formula. Behnke [2] studied this problem, with giving some asymptotic orders for the behaviors of the sum by using Fourier analysis.

The cancellations technique plays an important role in calculation of $\sum_{i=1}^{N} \left(\{i\alpha\} - \frac{1}{2} \right)$ in the previous subsection, because $\left(\{i\alpha\} - \frac{1}{2} \right)$ is linear in i. The cancellation technique, however, does not work for this problem, because $\left(\{i\alpha\} - \frac{1}{2} \right)^2$ is not linear. For this reason, we have to calculate the sum $\sum_{i=1}^{N} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$ directly.

Because of our exact result, we can show the fact that $\sum_{i=1}^{N} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$ behaves like a cubic function, when we observe it with an adequate steps. We first state our main results.

Theorem 4.3 (cf. [16])**.** *Let α be an irrational number, $0 < \alpha < 1$, and m be the number determined by Ostrowski expansion, for any natural number N.*

$$\sum_{i=1}^{N}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\sum_{j:even,\leq m}^{m}\left\{\frac{1}{3}b_j^3\Delta_j^2 q_j-\frac{1}{2}b_j^2(\Delta_j-\Delta_j^2-2s_j\Delta_j q_j)\right.$$

$$\left.+\frac{1}{6}b_j\Big(\frac{1}{q_j}+q_j\delta_j^2+6s_jq_j\delta_j-6s_j+6s_j^2q_j\Big)+2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_i}{q_j}-\frac{1}{2}b_jq_j^2\delta_j\right\}$$

$$+\sum_{j:odd,\leq m}^{m}\left\{\frac{1}{3}b_j^3\Delta_j^2q_j-\frac{1}{2}b_j^2(\Delta_j-\Delta_j^2+2s_j\Delta_jq_j)+\frac{1}{6}b_j\Big(\frac{1}{q_j}+q_j\delta_j^2-6s_jq_j\delta_j+6s_j+6s_j^2q_j\Big)\right.$$

$$\left.-2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_i}{q_j}+\frac{1}{2}b_jq_j^2\delta_j+b_jq_j\delta_j\right\},$$

where $\delta_j = \left|\alpha-\dfrac{p_j}{q_j}\right|$, $s_j = \displaystyle\sum_{k=j+1}^{m} b_kq_k\left(\alpha-\frac{p_k}{q_k}\right)$, $s_m = 0$, $\Delta_j = q_j\delta_j$, $ip_j \equiv k_i$ (mod q_j).

Proof. We first decompose the sum $\sum_{i=1}^{b_jq_j}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$ by Ostrowski expansion. We use the same notations n_j, s_j as those defined in previous subsection.

$$\begin{aligned}\sum_{i=1}^{N}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}&=\sum_{j=0}^{m}\sum_{i=n_j+1}^{n_j+b_jq_j}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}\\&=\sum_{j=0}^{m}\sum_{i=1}^{b_jq_j}\left\{\left(\{s_j+i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}\\&=\sum_{j=0}^{m}\sum_{i=1}^{b_jq_j}\left\{\left(s_j+\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}.\end{aligned}$$

Case I. We first assume that j is even.

Let us recall that $\{(i+\ell q_j)\alpha\} = \{i\alpha\}+\ell\Delta_j$ and $\{i\alpha\} = k_i/q_j + i\delta_j$ ($i = 1,\dots,q_j$, $\ell = 0,\dots,b_j-1$).

$$
\begin{aligned}
&\sum_{i=1}^{b_j q_j}\left\{\left(s_j+\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\} \\
&=\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(s_j+\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\frac{1}{12}\right\} \\
&=\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left(\left\{\left(s_j+\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2\right\}\right. \\
&+\left\{\left(\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2\right\} \\
&\left.+\left\{\left(i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(\frac{k_i}{q_j}-\frac{1}{2}\right)^2\right\}+\left\{\left(\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}\right).
\end{aligned}
$$

We now calculate each terms in this double summation, having the form of $\{(A+\varepsilon)^2-A^2\}$. We obtain the following the equations:

(i)

$$
\begin{aligned}
&\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(s_j+\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2\right\} \\
&=s_j b_j(b_j\Delta_j q_j-\Delta_j q_j+q_j^2\delta_j+q_j\delta_j-1+s_j q_j)\ ,
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(\ell\Delta_j+i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(i\delta_j+\frac{k_j}{q_j}-\frac{1}{2}\right)^2\right\} \\
&=\frac{1}{6}b_j(b_j-1)(2b_j-1)\Delta_j^2 q_j+\frac{1}{2}b_j(b_j-1)\Delta_j q_j^2\delta_j \\
&+\frac{1}{2}b_j(b_j-1)\Delta_j q_j\delta_j-\frac{1}{2}b_j(b_j-1)\Delta_j\ ,
\end{aligned}
$$

(iii)

$$\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\left(\frac{k_i}{q_j}-\frac{1}{2}\right)^2\right\}$$

$$=\frac{1}{3}b_jq_j^3\delta_j^2+\frac{1}{2}b_jq_j^2\delta_j^2+\frac{1}{6}b_jq_j\delta_j^2-\frac{1}{2}b_jq_j^2\delta_j-\frac{1}{2}b_jq_j\delta_j+2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_i}{q_j}\ ,$$

(iv)

$$\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(\frac{k_i}{q_j}\right)^2-\frac{k_i}{q_j}+\frac{1}{6}\right\}$$

$$=\frac{1}{6q_j}b_j\ .$$

Summing up the above calculations, we obtain the following result:

$$\sum_{i=1}^{b_jq_j}\left\{\left(s_j+\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\frac{1}{3}b_j^3\Delta_j^2q_j-\frac{1}{2}b_j^2\left(\Delta_j-\Delta_j^2-2s_j\Delta_jq_j\right)$$

$$+\frac{1}{6}b_j\left(\frac{1}{q_j}+q_j\delta_j^2+6s_jq_j\delta_j-6s_j+6s_j^2q_j\right)+2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_i}{q_j}-\frac{1}{2}b_jq_j^2\delta_j.$$

Case II. We next assume that j is odd.

In this case, note that $\{i\alpha\}-k_i/p_j=-i\delta_j$ and $\{(i+\ell q_j)\alpha\}=\{i\alpha\}-\ell\Delta_j$ ($i=1,\dots,q_j\ ,\ \ell=0,\dots,b_j-1$). Now we obtain the following sum by similar arguments which we use calculations for the case even j.

$$\sum_{i=1}^{b_jq_j}\left\{\left(s_j+\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\sum_{\ell=0}^{b_j-1}\sum_{i=1}^{q_j}\left\{\left(s_j-\ell\Delta_j-i\delta_j+\frac{k_i}{q_j}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$$

We similarly obtain the next estimate for odd j for even j:

$$\sum_{i=1}^{b_jq_j}\left\{\left(s_j+\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\frac{1}{3}b_j^3\Delta_j^2q_j-\frac{1}{2}b_j^2\left(\Delta_j-\Delta_j^2+2s_j\Delta_jq_j\right)$$
$$+\frac{1}{6}b_j\left(\frac{1}{q_j}+q_j\delta_j^2-6s_jq_j\delta_j+6s_j+6s_j^2q_j\right)-2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_i}{q_j}+\frac{1}{2}b_jq_j^2\delta_j+b_jq_j\delta_j.$$

Summing up the above calculations, we obtain the following formula:

$$\sum_{i=1}^{N}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\sum_{j:even,0\le j\le m}^{m}\left\{\frac{1}{3}b_j^3\Delta_j^2q_j-\frac{1}{2}b_j^2(\Delta_j-\Delta_j^2-2s_j\Delta_jq_j)\right.$$
$$\left.+\frac{1}{6}b_j\Big(\frac{1}{q_j}+q_j\delta_j^2+6s_jq_j\delta_j-6s_j+6s_j^2q_j\Big)+2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_j}{q_j}-\frac{1}{2}b_jq_j^2\delta_j\right\}$$
$$+\sum_{j:odd,0\le j\le m}^{m}\left\{\frac{1}{3}b_j^3\Delta_j^2q_j-\frac{1}{2}b_j^2(\Delta_j-\Delta_j^2+2s_j\Delta_jq_j)\right.$$
$$+\frac{1}{6}b_j\Big(\frac{1}{q_j}+q_j\delta_j^2-6s_jq_j\delta_j+6s_j+6s_j^2q_j\Big)$$
$$\left.-2b_j\delta_j\sum_{i=1}^{q_j}i\frac{k_j}{q_j}+\frac{1}{2}b_jq_j^2\delta_j+b_jq_j\delta_j\right\}.$$

□

Next let us consider the sum $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$ for specific value n, $n=\nu q_j$, for some j ($\le m$), then we obtain simpler formulas:

Corollary 4.4 (cf. [16])**.** *For $n=\nu q_j$, let $x_\nu=\frac{\nu}{a_{j+1}}$, for $0\le\nu\le a_{j+1}$.*

$$\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}=\frac{1}{3}a_{j+1}^3\Delta_j^2q_j\left\{x_\nu^3-\frac{3}{2}\left(\frac{1}{a_{j+1}\Delta_jq_j}-\frac{1}{a_{j+1}q_j}\right)x_\nu^2\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{a_{j+1}^2\Delta_j^2q_j^2}+\frac{1}{a_{j+1}^2q_j^2}\right)x_\nu\right\}+\theta,$$

where,

$$\theta=\begin{cases}2\nu\delta_j\sum_{i=1}^{q_j}i\dfrac{k_i}{q_j}-\dfrac{1}{2}\nu q_j^2\delta_j, & \textit{if } j \textit{ is even},\\ -2\nu\delta_j\sum_{i=1}^{q_j}i\dfrac{k_i}{q_j}+\dfrac{1}{2}\nu q_j^2\delta_j+\nu q_j\delta_j & \textit{if } j \textit{ is odd}.\end{cases}$$

When the continued fraction expansion of α has an isolated large partial quotient a_{j+1}, especially irrational number α can be closely approximated by jth convergent r_j. Hence we can easily see that $\frac{1}{a_{j+1}\Delta_j q_j}$ almost equals 1 and $\frac{1}{a_{j+1}q_j}$ is very small. Then the right-hand side of this equation seems to be a cubic function of $\nu, 0 \leq \nu \leq a_{j+1}$. Thus we can give an explanation for the unusual behaviors of the sum $\sum_{i=1}^{\nu q_j}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$, which are shown in graphs, Fig.3 and Fig.4.

Corollary 4.5 (cf. [16]). *For* $0 \leq \nu < a_{j+1}$.

$$\sum_{i=1}^{\nu q_j}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\} = \frac{a_{j+1}}{3q_j}(1+\varepsilon_1)x_\nu\left(x_\nu-\frac{1}{2}\right)(x_\nu-1)+\theta+\varepsilon_2\ ,$$

where $x_\nu = \nu/a_{j+1}\ ,\ 0 \leq x_\nu < 1\ ,\ |\varepsilon_1| < \dfrac{1}{q_j}$ *and* $|\varepsilon_2| < \dfrac{7}{3q_j}$.

4.3. Some examples

First, let us recall the following continued fraction expansions:

$$\pi - 3 = [0; 7, 15, 1, 292, 1, 1, 1, 2, \ldots],$$

and

$$\log_{10} 7 = [0; 1, 5, 2, 5, 6, 1, 4813, 1, 1, \ldots].$$

We discuss the behaviors of irrational rotations based on these irrational numbers. Our arguments are, however, still valid with respect to irrational rotations based on similar irrational numbers with large partial quotients.

4.3.1. The behaviors of $\sum_{i=1}^{n}\left(\{i\alpha\}-\frac{1}{2}\right)$

We discuss with the effects on the sums $\sum_{i=1}^{n}\left(\{i\alpha\}-\frac{1}{2}\right)$, caused by large partial quotients, that is, 4813 for $\log_{10} 7$, and 292 for $\pi - 3$. We consider the partial sums for $n = \nu q_j$, for $0 \leq \nu < a_{j+1}$. By Corollary 4.2, the partial sum $\sum_{i=1}^{\nu q_j}\left(\{i\alpha\}-\frac{1}{2}\right)$

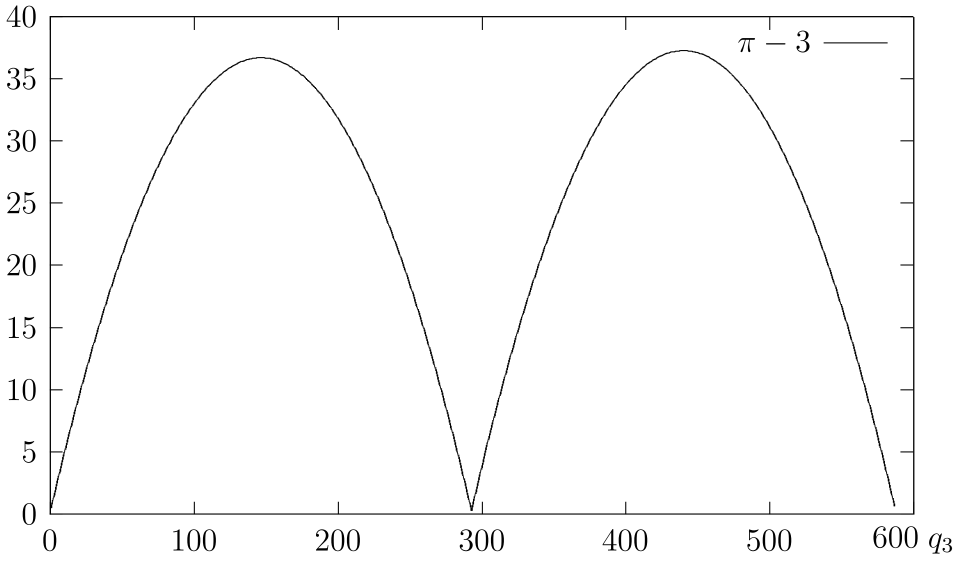
can be approximated by

$$(-1)^j \frac{a_{j+1}}{2} \frac{\nu}{a_{j+1}} \left(\frac{\nu}{a_{j+1}} - 1 \right) = (-1)^j \frac{a_{j+1}}{2} x_\nu (x_\nu - 1),$$

where $x_\nu = \nu / a_{j+1}$, $0 < x_\nu < 1$, since the error terms ε_1 and ε_2 are negligible.

For example, in the case of $\alpha = \pi - 3$, the maximum value of the curve of the values of the sum is almost equal to $a_4/8$. Similarly, in the case of $\alpha = \log_{10} 7$, the minimum value of the curve is almost equal to $-a_7/8$ (cf. Fig.1 and Fig.2).

Fig.1. $\sum_{i=1}^{N} \left(\{i\alpha\} - \frac{1}{2} \right)$, $\alpha = \pi - 3$, $n = 587 \times 113$



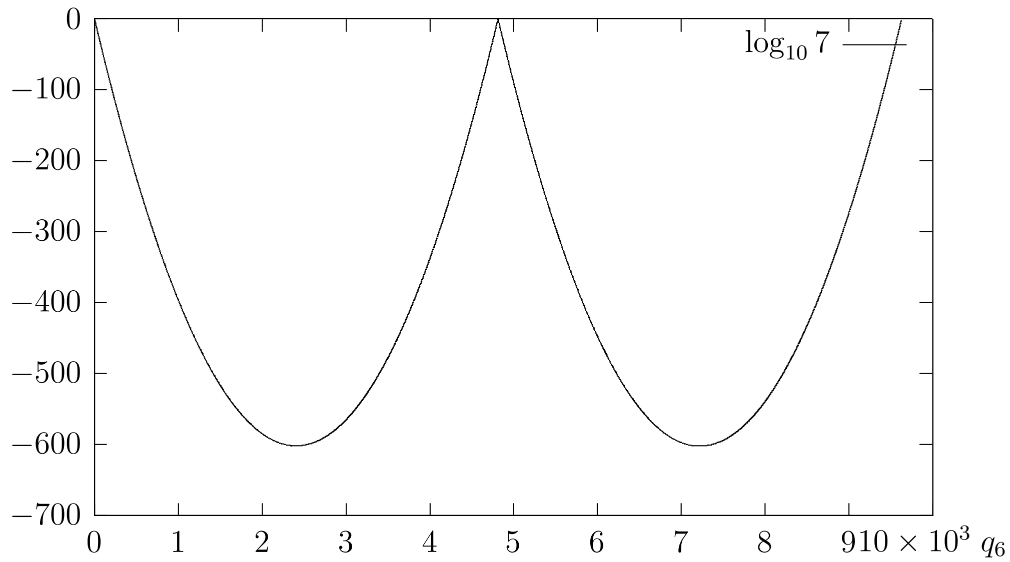
4.3.2. The behaviors of $\sum_{i=1}^{n} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$

By Corollary 4.5, if $\alpha = \pi - 3$, or $\log_{10} 7$, the error term are also negligible, and the partial sum $\sum_{i=1}^{\nu q_j} \left\{ \left(\{i\alpha\} - \frac{1}{2} \right)^2 - \frac{1}{12} \right\}$ can be approximated by the following cubic function:

$$\frac{a_{j+1}}{3q_j} x_\nu \left(x_\nu - \frac{1}{2} \right) (x_\nu - 1),$$

where $x_\nu = \nu / a_{j+1}$, $0 \le x_\nu < 1$.

Fig.2. $\sum_{i=1}^{N}\left(\{i\alpha\}-\frac{1}{2}\right)$, $\alpha=\log_{10}7$, $n=9627\times=510$



For $\alpha=\pi-3$, the first maximal value of the curve is almost equal to $\frac{\sqrt{3}a_4}{108q_3}$, and the first minimal value of the curve is almost equal to $-\frac{\sqrt{3}a_4}{108q_3}$. (cf. Fig. 3)

For $\alpha=\log_{10}7$, the first maximal value of the curve is almost equal to $\frac{\sqrt{3}a_7}{108q_6}$, and the first minimal value of the curve is almost equal to $-\frac{\sqrt{3}a_7}{108q_6}$. (cf. Fig. 4)

By the way, in the discrepancy graph of irrational rotations, Setokuchi and Takashima [23], and Setokuchi [22] investigated repetitions of "hills" and "valleys", when α has a single isolated large partial quotient. For the partial sum $\sum_{i=1}^{n}\left(\{i\alpha\}-\frac{1}{2}\right)$, as we showed in subsection 4.1, we can also observe a similar "periodic" behavior for $\alpha=\pi-3$ and for $\log_{10}7$, see Fig.1 and Fig.2.

As for the behavior of the partial sum $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$, in the case of $\pi-3$ we observe very different phenomena, see Fig.4. Note that in this case we do not deal with simple repetitions. Schoissengeier proved the following approximation:

Theorem A ([21] Theorem 3, pp.136)**.**

$$\sum_{i=1}^{N}B_2(\{i\alpha\})=\frac{1}{3}\sum_{t=0}^{\infty}B_3\Big(\frac{b_t}{a_{t+1}}\Big)\frac{a_{t+1}}{q_t}+O(1),$$

where $B_2(x)$ and $B_3(x)$ are Bernoulli polynomials.

Note that the sum $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$ itself is $O(1)$ and thus Theorem A does not explain Fig.5 and Fig.6: to this purpose we need to show that the error term $O(1)$ on the right hand side is small compared with the sum, as shown by Corollaries 4.4 as 4.5 above.

Fig.3. $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$, $\alpha=\pi-3,\quad n=584\times 113$,

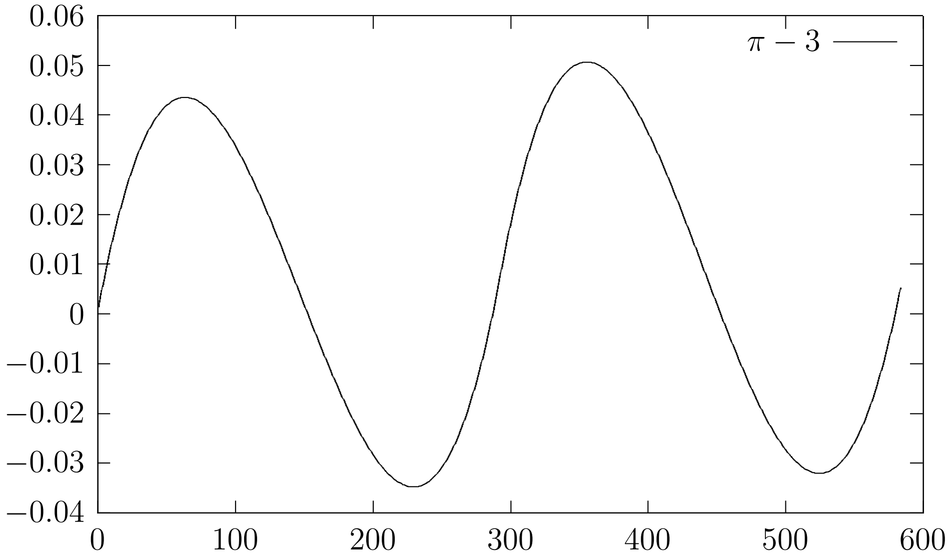


Fig.4. $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$, $\alpha=\log_{10}7$, $n=9627\times 510$,

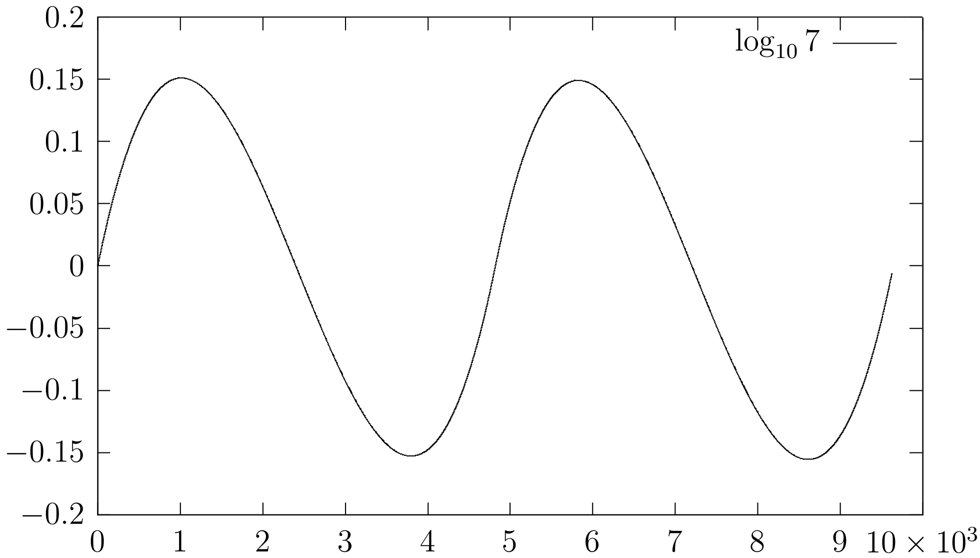


Fig.5. $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$ $\alpha=\pi-3$, $n=2920\times 113$,

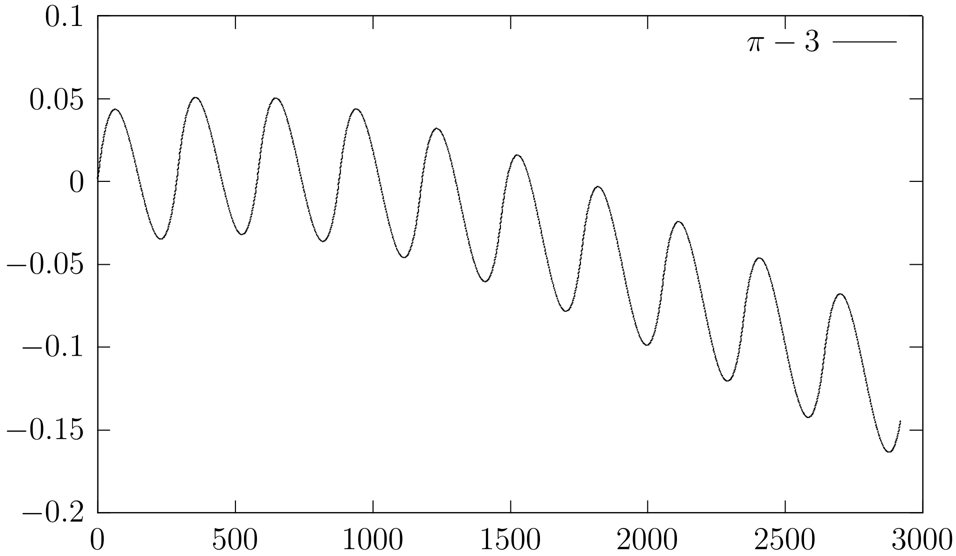
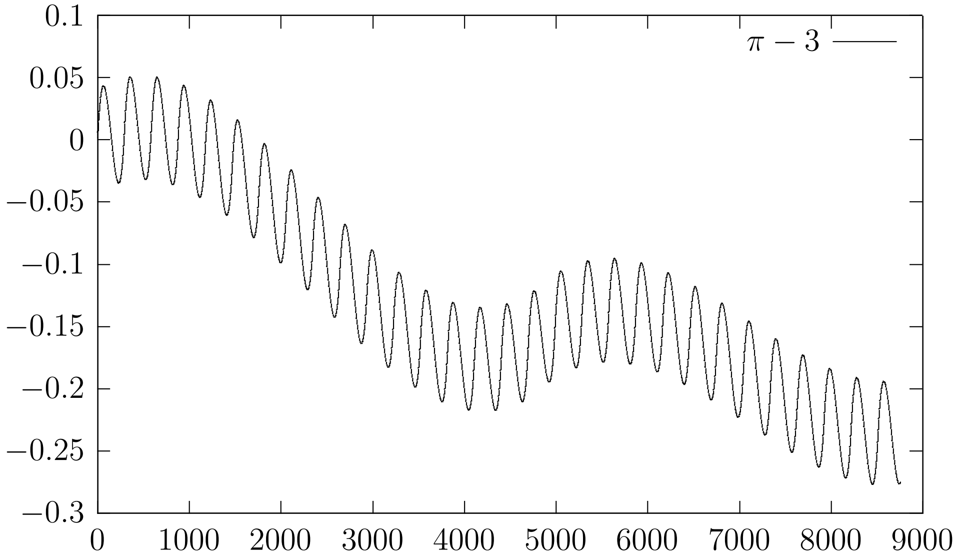


Fig.6. $\sum_{i=1}^{n}\left\{\left(\{i\alpha\}-\frac{1}{2}\right)^2-\frac{1}{12}\right\}$ $\alpha=\pi-3$, $n=8760\times 113$,



5. The estimate of discrepancy

5.1. Another proof of Weyl's lemma

We give an another proof of Weyl's lemma by applying rational rotation approximation, Ostrowski expansion and basic properties of discrepancies. We decompose the sum in the definition of $ND_N^*(\{i\alpha\})$ by using Ostrowski expansion as follow:

$$\begin{aligned}\sum_{i=1}^{N} \mathbf{I}_{[0,x)}(\{i\alpha\}) - Nx &= \sum_{j=0}^{m} \left(\sum_{i=n_j+1}^{n_j+b_jq_j} \mathbf{I}_{[0,x)}(\{i\alpha\}) - b_jq_jx \right) \\ &= \sum_{j=0}^{m} \left(\sum_{i=1}^{b_jq_j} \mathbf{I}_{[0,x)}(s_j + \{i\alpha\}) - b_jq_jx \right).\end{aligned}$$

Here, we consider the estimate for $\sum_{j=0}^{m} \left(\sum_{i=1}^{b_jq_j} \mathbf{I}_{[0,x)}(s_j + \{i\alpha\}) - b_jq_jx \right)$. It is easily seen that each term in the first summation of the right-hand side is almost equal to a corresponding term in the definition of $b_jq_jD^*_{b_jq_j}$, except being shifted just by s_j.

From the fact that each sub-interval, $(k/q_j,\ (k+1)/q_j)$, contains just b_j points of $\{i\alpha\}$, $i = 1, 2, \ldots, b_jq_j$, $b_j < a_{j+1}$. We easily obtain the following estimate:

$$\left| \sum_{i=1}^{b_jq_j} \mathbf{I}_{[0,x)}(s_j + \{i\alpha\}) - b_jq_jx \right| \leq 2a_{j+1}.$$

Similarly we have

$$\left| \sum_{i=1}^{b_mq_m} \mathbf{I}_{[0,x)}(\{i\alpha\}) - b_mq_mx \right| \leq 2b_m.$$

Summing up these arguments, we have an estimate for ND_N^* from above:

$$ND_N^* \leq 2\left(\sum_{j=0}^{m-1} a_{j+1} + b_m \right).$$

Although Khinchine [13] showed that $q_n \geq 2^{\frac{k-1}{2}}$, we provide A $(= \max_{1\leq j\leq m} a_j)$ to remove the possibility that a_j is large. Then, the following estimates are easily derived;

$$N \geq A2^{\frac{m-1}{2}}\ (m > 1),\ \sum_{j=0}^{m-1} a_{j+1} \leq Am,\ \frac{b_m}{N} < \frac{1}{q_m},$$

and

$$D_N^*(\{i\alpha\}) \leq m2^{\frac{-m+3}{2}} + \frac{2}{q_m}.$$

Thus, we easily obtain $\lim_{N\to\infty} D_N^*(\{i\alpha\}) = 0$, and this prove Weyl's lemma.

5.2. **The estimate of discrepancy: single isolated large partial quotient case**

We discuss the behaviors of discrepancies of irrational rotations with single isolated large partial quotient. It is well-known that $\pi - 3$ has a single isolated large partial quotient $a_\eta = a_4 = 292$. For example, Fig.7 shows the first slope and the first "hill" of a graph of discrepancies of irrational rotation based $\alpha = \pi - 3$.

5.2.1. **Estimate for the first slope**

We, first, consider $n = \nu \times q_{\eta-1}$, $\nu = 1, 2, \ldots, a_\eta - 1$, and thus, $i < q_\eta$. Behaviors of discrepancies for these i's are shown as the first long slope in graphs, cf. Fig.7 and Fig.8. For such n's, $\{i\alpha\}$ is very closely approximated by $\left\{i\frac{p_j}{q_j}\right\}$.

We obtain the following estimate for the first part of graphs of discrepancies. In the following, we assume that the order η of the isolated large partial quotient is odd and $\eta \geq 3$. In case where η is even, we can easily change our arguments in the reverse direction. Then, we have:

Theorem 5.1. *Assume that $\eta \geq 3$. For $n = \nu q_{\eta-1}(0 < \nu < a_\eta)$, $D_n^*(\{i\alpha\})$ is linearly decreasing, i.e.,*

$$D_n^*(\{i\alpha\}) = \frac{1}{q_{\eta-1}}\left(1 - \Delta_{\eta-1} - \frac{(\nu-1)}{L}\right),$$

where $L = q_{\eta-1}\Delta_{\eta-1} = a_\eta + \frac{q_{\eta-2}}{q_{\eta-1}} + \tau^\eta(\alpha)$. We also have

$$nD_n^*(\{i\alpha\}) = \nu\left(1 - \Delta_{\eta-1} - \frac{(\nu-1)}{L}\right).$$

where $\Delta_{\eta-1} = q_{\eta-1}|\alpha - r_{\eta-1}|$ and $L = a_\eta + \frac{q_{\eta-2}}{q_{\eta-1}} + \tau^\eta(\alpha)$.

Let us recall that the point $\{(i+q_{\eta-1})\alpha\}$ returns back into the same interval where the point $\{i\alpha\}$ lies, for any $i < a_\eta$, just greater by $\Delta_{\eta-1}$, we can estimate $D_n^*(\{i\alpha\})$ as follows: let $n = \nu\, q_{\eta-1}$, first we can divide the supremum into small parts;

Fig.7. $\alpha = \pi - 3$ with step $q_3 = 113$

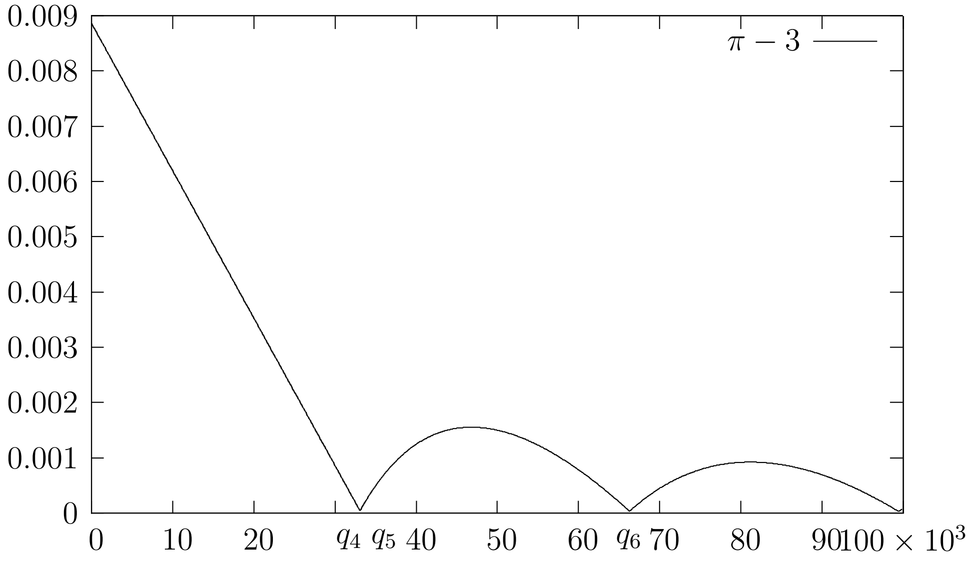
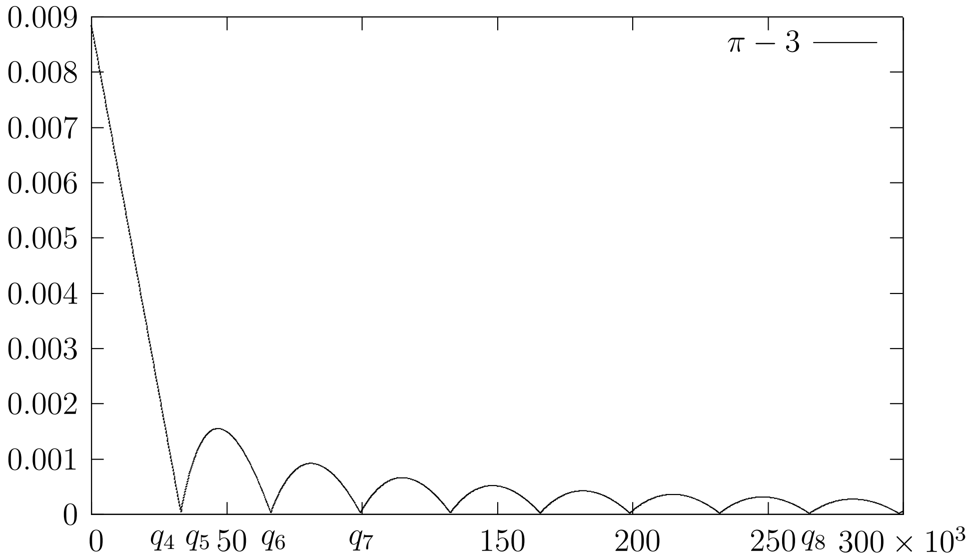


Fig.8. $\alpha = \pi - 3$ with step $q_3 = 113$



$$n\, D_n^*(\{i\alpha\}) = \sup_{0 \le i < q_{\eta-1}} \sup_{k_i \le q_{\eta-1} x < (k_i+1)} \left| \sum_{i=1}^{n} I_{[0,x)}(\{i\alpha\}) - nx \right| .$$

Note here that $\left|\sum_{i=1}^{n} I_{[0,x_{k_i})}(\{i\alpha\}) - nx_{k_i}\right| = 0$, for any $x_{k_i} = \frac{k}{q_{\eta-1}}$, $ip_{\eta-1} \equiv k_i$ (mod $q_{\eta-1}$) and for $n = \nu \times q_{\eta-1}$, because *just* ν points of $\{i\alpha\}$ are in each interval $[x_{k_i}, x_{k_i+1})$. It is also easily proved that the supremum of the value in the right-hand side is attained at the largest point $i\alpha \pmod 1$ in the interval $[x_{k_i}, x_{k_i+1})$. Thus, we have

$$\sup_{x_{k_i} \le x < x_{k_i+1}} \left| \sum_{i=1}^{n} I_{[x_{k_i},x)}(\{i\alpha\}) - nx \right| = \nu - \nu q_{\eta-1}(i\delta_{\eta-1} + (\nu-1)\Delta_{\eta-1}),$$

where $\delta_{\eta-1} = |\alpha - r_{\eta-1}|$. Thus, we have the following estimate:

$$nD_n^*(\{i\alpha\}) = \nu - \nu q_{\eta-1}(\delta_{\eta-1} + (\nu-1)\Delta_{\eta-1}) = \nu - \nu(q_{\eta-1}\delta_{\eta-1} + (\nu-1)q_{\eta-1}\Delta_{\eta-1}),$$

for $n = \nu q_{\eta-1}$, $\nu < a_\eta$. We use that $\Delta_{\eta-1} = q_{\eta-1}\delta_{\eta-1}$ and $q_{\eta-1}\Delta_{\eta-1} = \frac{1}{L}$, then we can obtain the following:

$$nD_n^*(\{i\alpha\}) = \nu \left(1 - (\Delta_{\eta-1} + \frac{(\nu-1)}{L})\right),$$

and

$$D_n^*(\{i\alpha\}) = \frac{1}{q_{\eta-1}} \left(1 - (\Delta_{\eta-1} + \frac{(\nu-1)}{L})\right),$$

for $n = \nu q_{\eta-1}$, $\nu < a_\eta$.

In case where η is even, we can derive the result by considering the direction reversed.

5.2.2. **Estimate for the first hill**

Setokuchi ([22]) (see also [24]) shows that the first valley is a wider valley, $[q_\eta, q_{\eta+1}]$ in case of $a_{\eta+1} = 1$, and that the first valley is a one-point valley, $[q_\eta, q_\eta]$ in case of $a_{\eta+1} \ge 2$.

The basic idea of our calculations in this subsection is that hills between valleys, one-point valleys or wider valleys, have width q_η and that at the starting point of each hill, the discrepancy up to the point is evaluated by the summation of discrepancies for hills involved. We can say this as follows; *discrepancies can start afresh again at each end-point of a valley.*

To fix our consideration, we assume first that η is odd and that $a_{\eta+1} = 1$. Then, the first valley is a wider valley and the first hill starts from $n = q_{\eta+1}+1$. Recall that every point $\{i\alpha\}$ $(1 \le i \le q_{\eta+1})$ is approximated by some $\frac{j}{q_{\eta+1}}$ $(j = 0, 1, \ldots, q_{\eta+1}-1)$

very highly closely.

Next, we consider the behaviors of discrepancies of irrational rotations for $n = q_{\eta+1} + \nu q_{\eta-1}\,(0 < \nu < a_\eta)$. The orbit $\{(q_{\eta+1}+k)\alpha\}$ $(k \geq 0)$ is almost equal to the orbit $\{k\alpha\}$ $(k \geq 0)$. The difference is that each point is just shifted by δ', where $\delta' = (\alpha - r_{\eta+1})q_{\eta+1} = q_{\eta+1}\alpha - p_{\eta+1} \pmod 1$. Note that $|\delta'| < \frac{1}{q_{\eta+2}}$. Moreover, since

$$\sum_{i=1}^{n} I_{[0,x)}(\{i\alpha\}) - nx$$

is linear, we can divide this into two parts; one is concerned with only points $\{i\alpha\}$ $(1 \leq i \leq q_{\eta+1})$, and the other is concerned with other points $\{m\alpha\}$ $(q_{\eta+1}+1 \leq m \leq q_{\eta+1} + \nu q_{\eta-1})$. Thus we easily have

$$\sum_{m=1}^{q_{\eta+1}+\nu q_{\eta-1}} I_{[0,x)}(\{i\alpha\}) - (q_{\eta+1} + \nu q_{\eta-1})x$$

$$= \left(\sum_{i=1}^{q_{\eta+1}} I_{[0,x)}(\{m\alpha\}) - q_{\eta+1}x \right) + \left(\sum_{i=q_{\eta+1}+1}^{q_{\eta+1}+\nu q_{\eta-1}} I_{[0,x)}(\{i\alpha\}) - \nu q_{\eta-1}x \right).$$

The first term is exactly related to the discrepancy of points $\{i\alpha\}$ $(1 \leq i \leq q_{\eta+1})$, and the second term is divided into two parts:

$$\sum_{i=q_{\eta+1}+1}^{q_{\eta+1}+\nu q_{\eta-1}} I_{[0,x)}(\{i\alpha\}) - \nu q_{\eta-1}x$$

$$= \left(\sum_{i=q_{\eta+1}+1}^{q_{\eta+1}+\nu q_{\eta-1}} I_{[0,x)}(\{i\alpha\}) - \nu q_{\eta-1}(x+\delta') \right) + \delta' \nu q_{\eta-1}.$$

$$= \left(\sum_{i=1}^{\nu q_{\eta-1}} I_{[0,x)}(\delta' + \{i\alpha\}) - \nu q_{\eta-1}(x+\delta') \right) + \delta' \nu q_{\eta-1}.$$

Note that points $\{(q_{\eta+1}+k)\alpha\}$ $(k \geq 0)$ are just shifted by δ' from points $\{k\alpha\}$ $(k \geq 0)$, and we can see that the first term of the right-hand side in the above last equation is the same as the summation discussed in the previous subsection.

Note that $q_{\eta+1}D^*_{q_{\eta+1}}(\{i\alpha\}) < 1$, and that $\eta - 1$ and $\eta + 1$ are even or odd simultaneously.

In case $a_{\eta+1} \geq 2$, then the point $n = q_\eta$ is a one-point valley, and the first hill represents the behaviors of $nD_n^*(\{i\alpha\}), q_\eta \leq i < 2q_\eta$. In this case, since η is odd, it is well-known that $\delta'' = q_\eta(\alpha - r_\eta) = q_\eta\alpha - p_\eta < 0$. Thus, we have to modify the above arguments a little, with changing δ' by δ''.

In case of even η, we need only to consider the reverse direction. Thus, we can easily obtain the following theorem:

Theorem 5.2. *Assume that $\eta \geq 3$. We have the following estimate for the first hill:*

$$nD_n^*(\{i\alpha\}) = \nu\left(1 - \Delta_{\eta-1} - \frac{(\nu-1)}{L}\right) + \theta,$$

for $n = q_{\eta+1} + \nu q_{\eta-1}$, if $a_{\eta+1} = 1$, and for $n = q_\eta + \nu q_{\eta-1}$, if $a_{\eta+1} \geq 2$, and for $0 < \nu < q_\eta$, where $0 \leq \theta \leq 2$.

Remark 5.1. From this theorem, it is clear that nD_n^* behaves like quadratic function, as remarked in Setokuchi and Takashima ([24]) and Setokuchi ([22]).

We can easily derive the following results from the above arguments:

Corollary 5.3. *Assume that $\eta \geq 3$ and $\lambda > \eta$.*
For $n = q_\lambda + \nu q_{\eta-1}$, $0 < \nu < a_\eta$,

$$nD_n^*(\{i\alpha\}) = \nu\left(1 - \Delta_{\eta-1} - \frac{(\nu-1)}{L}\right) + \theta,$$

where $0 \leq \theta \leq 2$.

Corollary 5.4. *The hills, caused by single isolated large partial quotient, q_η, will appear infinitely often.*

In the above, we assume that $a_{\eta+1}$ is not large compared with a_η. In case of *large* $a_{\eta+1}$, which we can not call it as the case of *single isolated* large partial quotient, we must be very careful in estimations of discrepancies.

5.3. **The estimate of discrepancy: several large partial quotients case**

We consider discrepancies of irrational rotations based on an irrational number α, and we pick out several specific partial quotients $a_{\eta_1}, a_{\eta_2}, \ldots, a_{\eta_K}$ from α's continued fraction expansion. We investigate the effects of $a_{\eta_1}, a_{\eta_2}, \ldots, a_{\eta_K}$ on the behaviors of discrepancies. Then, we have

Theorem 5.5. *Let α be an irrational number, $0 < \alpha < 1$. For any positive integer N, m denotes the positive integer defined in Ostrowski expansion.*

$$\Bigg| ND_N^*(\{i\alpha\}) - \max\Bigg\{ \sum_{\eta_k:odd} \{b_{\eta_k-1} - b_{\eta_k-1}q_{\eta_k-1}\left(s_{\eta_k-1} + \delta_{\eta_k-1} + (b_{\eta_k-1}-1)\Delta_{\eta_k-1}\right)\},$$

$$\sum_{\eta_k:even} \{b_{\eta_k-1} - b_{\eta_k-1}q_{\eta_k-1}\left(s_{\eta_k-1} + \delta_{\eta_k-1} + (b_{\eta_k-1}-1)\Delta_{\eta_k-1}\right)\} \Bigg\}\Bigg|$$

$$\leq \max\{K_1, K_2\} + \frac{1}{4}\max\left\{ \sum_{j=0, j\neq \eta_1,\cdots,\eta_K, j:even}^{m} \overline{a}_j, \sum_{j=0, j\neq \eta_1,\cdots,\eta_K, j:odd}^{m} \overline{a}_j \right\},$$

where $\overline{a}_j = \max\{4, a_j\}$, K_1 denotes the number of even $\eta_k{}'s$ and K_2 denotes the number of odd $\eta_k{}'s$. Moreover let $n_j = \sum_{k=j+1}^{m} b_k q_k$, $s_j = \sum_{k=j+1}^{m} b_k q_k(\alpha - r_k)$, $j = m-1, \ldots, 0$, and $s_m = 0$.

Remark 5.2. When several specific partial quotients $a_{\eta_1}, a_{\eta_2}, \ldots, a_{\eta_K}$ are comparatively large, the terms in the right–hand side in Theorem 5.5 are much smaller than the maximum in the left–hand side. Therefore, discrepancies of such irrational rotations behave according to the maximum in the left–hand side.

Remark 5.3. When we choose more specific partial quotients, the number of members in the sums in the right–hand side would be smaller. Contrary, when we choose less specific partial quotients, the number of members in the sums in the right–hand side would be bigger.

Proof. We decompose the sum in the definition of $nD_n^*(\{i\alpha\})$ by using Ostrowski expansion, as follows:

$$\begin{aligned}\sum_{i=1}^{n} \mathbf{1}_{[0,a)}(\{i\alpha\}) - Na &= \sum_{\eta_k:odd}\left(\sum_{i=1}^{b_{\eta_k-1}q_{\eta_k-1}} \mathbf{1}_{[0,a)}(s_{\eta_k-1}+\{i\alpha\}) - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+a)\right)\\ &+ \sum_{\eta_k:even}\left(\sum_{i=1}^{b_{\eta_k-1}q_{\eta_k-1}} \mathbf{1}_{[0,a)}(s_{\eta_k-1}+\{i\alpha\}) - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+a)\right)\\ &+ \sum_{\eta_k:odd}(b_{\eta_k-1}q_{\eta_k-1}s_{\eta_k-1}) + \sum_{\eta_k:even}(b_{\eta_k-1}q_{\eta_k-1}s_{\eta_k-1})\\ &+ \sum_{j=0, j\neq\eta_1-1,\cdots,\eta_K-1, j:odd}^{m}\left(\sum_{i=1}^{b_jq_j}\mathbf{1}_{[0,a)}(\{i\alpha\}) - b_jq_ja\right)\\ &+ \sum_{j=0, j\neq\eta_1-1,\cdots,\eta_K-1, j:even}^{m}\left(\sum_{i=1}^{b_jq_j}\mathbf{1}_{[0,a)}(\{i\alpha\}) - b_jq_ja\right).\end{aligned}$$

Using Lemma 2.12 and the well-known inequality $p_{2n}/q_{2n} < \alpha < p_{2n-1}/q_{2n-1}$, we have the following estimates for odd η_k $(\eta_k < m)$:

$$\begin{aligned}-1 &\leq \sum_{i=1}^{b_{\eta_k-1}q_{\eta_k-1}} \mathbf{1}_{[0,a)}(s_{\eta_k-1}+\{i\alpha\}) - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+a)\\ &\leq b_{\eta_k-1} - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+\delta_{\eta_k-1}+(b_{\eta_k-1}-1)\Delta_{\eta_k-1}),\end{aligned}$$

and for even η_k $(\eta_k < m)$:

$$\begin{aligned}&-\{b_{\eta_k-1} - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+\delta_{\eta_k-1}+(b_{\eta_k-1}-1)\Delta_{\eta_k-1})\} \leq\\ &\sum_{i=1}^{b_{\eta_k-1}q_{\eta_k-1}} \mathbf{1}_{[0,a)}(s_{\eta_k-1}+\{i\alpha\}) - b_{\eta_k-1}q_{\eta_k-1}(s_{\eta_k-1}+a) \leq 1.\end{aligned}$$

Let us recall that $|s_j| < \frac{1}{q_{j+1}}$, and it is easily shown that $|b_jq_js_j| < 1$. Then, we can obtain

$$\left|\sum_{\eta_k:odd}(b_{\eta_k-1}q_{\eta_k-1}s_{\eta_k-1}) + \sum_{\eta_k:even}(b_{\eta_k-1}q_{\eta_k-1}s_{\eta_k-1})\right| < \max\{K_1, K_2\},$$

where K_1 is the number of even $\eta_k's$ and K_2 is the number of odd $\eta_k's$. Set $\bar{a}_j = \max\{4, a_j\}$, then we have easily

$$\sup_{0\le a<1}\left|\sum_{j=0,j\neq\eta_1-1,\cdots,\eta_K-1,j:odd}^{m}\left(\sum_{i=1}^{b_jq_j}\mathbf{1}_{[0,a)}(\{i\alpha\})-b_jq_ja\right)\right.$$
$$\left.+\sum_{j=0,j\neq\eta_1-1,\cdots,\eta_K-1,j:even}^{m}\left(\sum_{i=1}^{b_jq_j}\mathbf{1}_{[0,a)}(\{i\alpha\})-b_jq_ja\right)\right|$$
$$\le\frac{1}{4}\max\left\{\sum_{j=0,j\neq\eta_1-1,\cdots,\eta_K-1,j:odd}^{m}\bar{a}_j,\sum_{j=0,j\neq\eta_1-1,\cdots,\eta_K-1,j:even}^{m}\bar{a}_j\right\}.$$

Summing up the above estimates, we obtain Theorem 5.5. □

When α has more large partial quotients $a_{\eta_1}, a_{\eta_2}, \ldots, a_{\eta_{K'}}$, the behaviors of $ND_N^*(\{i\alpha\})$ would become more complicated. Thus, we consider simpler cases when α has only two larger partial quotients a_{η_1}, a_{η_2}. Let us study behaviors of discrepancies in the case of even $\eta_1+\eta_2$ and in the case of odd $\eta_1+\eta_2$, separately. Firstly:

Corollary 5.6. *Assume that $\eta_1+\eta_2$ is even. For $n=\nu_1q_{\eta_1-1}+\nu_2q_{\eta_2-1}$ $(0\le\nu_1<a_{\eta_1}, 0\le\nu_2<a_{\eta_2})$,*

$$nD_n^*(\{i\alpha\})=\{\nu_1-\nu_1q_{\eta_1-1}\left(s_{\eta_1-1}+\delta_{\eta_1-1}+(\nu_1-1)\Delta_{\eta_1-1}\right)\}$$
$$+\{\nu_2-\nu_2q_{\eta_2-1}\left(s_{\eta_2-1}+\delta_{\eta_2-1}+(\nu_2-1)\Delta_{\eta_2-1}\right)\}+\theta,$$

where $|\theta|\le 2+\dfrac{1}{4}\max\left\{\sum_{j=0,j\neq\eta_1,\eta_2,j:even}^{m}\bar{a}_j,\sum_{j=0,j\neq\eta_1,\eta_2,j:odd}^{m}\bar{a}_j\right\}$.

Secondly:

Corollary 5.7. *Assume that $\eta_1+\eta_2$ is odd. For $n=\nu_1q_{\eta_1-1}+\nu_2q_{\eta_2-1}$ $(0\le\nu_1<a_{\eta_1}, 0\le\nu_2<a_{\eta_2})$,*

$$nD_n^*(\{i\alpha\})=\max\Big\{\{\nu_1-\nu_1q_{\eta_1-1}\left(s_{\eta_1-1}+\delta_{\eta_1-1}+(\nu_1-1)\Delta_{\eta_1-1}\right)\},$$
$$\{\nu_2-\nu_2q_{\eta_2-1}\left(s_{\eta_2-1}+\delta_{\eta_2-1}+(\nu_2-1)\Delta_{\eta_2-1}\right)\}\Big\}+\theta,$$

where $|\theta|\le 1+\dfrac{1}{4}\max\left\{\sum_{j=0,j\neq\eta_1,\eta_2,j:even}^{m}\bar{a}_j,\sum_{j=0,j\neq\eta_1,\eta_2,j:odd}^{m}\bar{a}_j\right\}$.

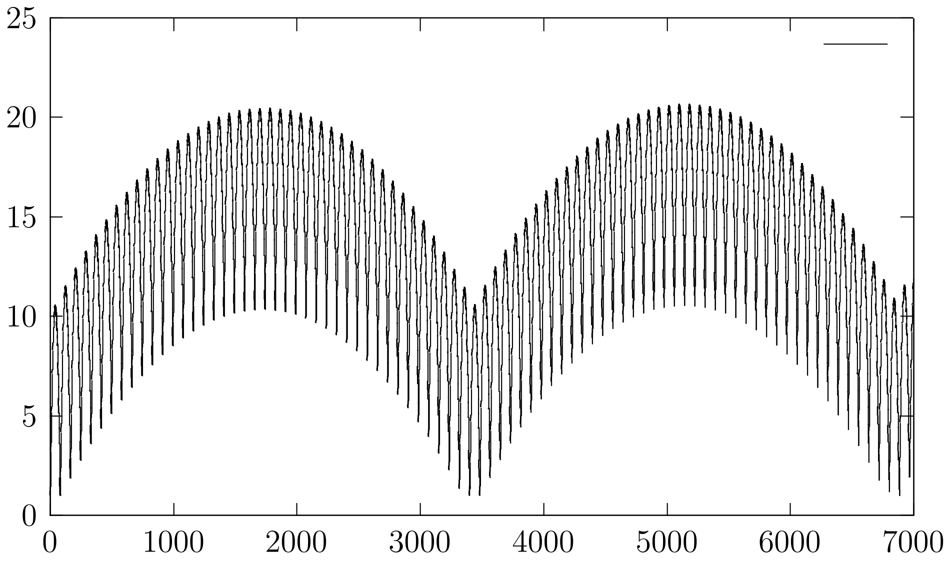
From Corollary 5.6 and Corollary 5.7, it can be easily seen that the shapes of graphs of discrepancies of irrational rotations with double large partial quotients would be quite different according to even $\eta_1+\eta_2$ or odd $\eta_1+\eta_2$ (see Fig.1, Fig.2).

5.4. **Some examples**

We consider the following irrational numbers $\alpha_1 = [\ 0; 2, 40, 1, 40, 2, 2, \dots\ ]$, $\alpha_2 = [\ 0; 2, 40, 40, 2, 2, \dots\ ]$ and $\alpha_3 = [0; 2, 1, 40, 20, 1, 10, 2, 2, \dots]$ and we observe behaviors of discrepancies of irrational rotations based on α_1, α_2 and α_3, because numerical calculations for these irrational numbers are comparatively easier. Note that α_1 is an example of even $\eta_1+\eta_2$ and α_2 is an example of odd $\eta_1+\eta_2$.

These three types of irrational rotations show typical unusual behaviors of discrepancies. By taking this into account, we give mathematical observations for discrepancies of irrational rotations based on $\pi-3$, $\log_{10} 7$ and $\log_{10} 37-1$.

Fig.9 $\quad \alpha_1 = [\ 0; 2, 40, 1, 40, 2, 2, \dots\ ]$

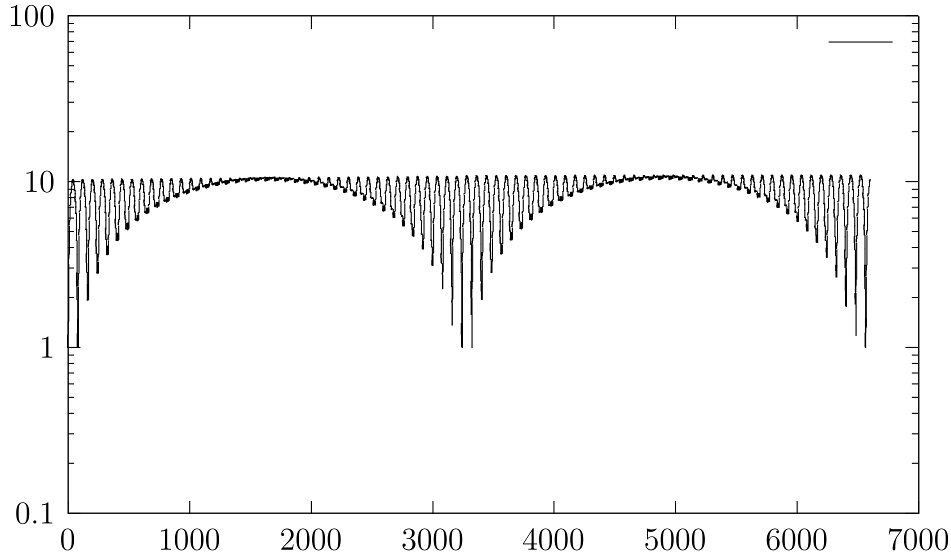


Example 5.8. Fig.9 shows the graph of $nD_n^*(\{i\alpha_1\})$. α_1 has double large partial quotient a_2, a_4. By applying Corollary 5.6, we can explain that 41 hills of short period q_2 are overlapped on one large hill of period q_4 because $q_4 = 41q_2 + 2$ (see

Fig.1). In the case of odd–order double large partial quotients, from Corollary 5.6, we can easily see overlapped hills like Fig.9.

Remark 5.4. In the case of $\alpha = \pi - 3$, α has double large partial quotients a_4 and a_{78}. Because 4 and 78 are both even, by taking into account the above considerations in Example 5.8, we see that hills of short period q_4 are overlapped on one large hill of period q_{78}, and we can expect overlapped hills like Fig.9.

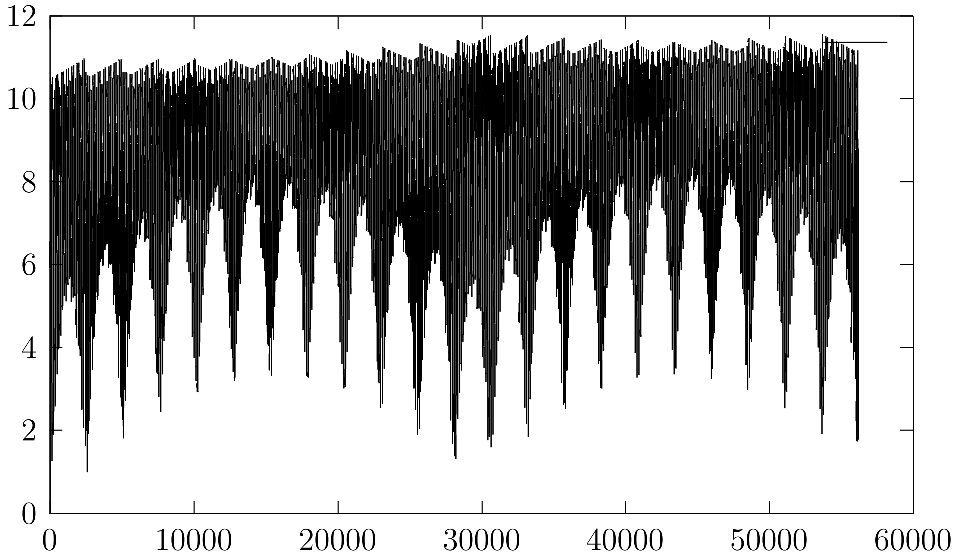
Fig.10 $\alpha_2 = [\,0; 2, 40, 40, 2, 2, \dots\,]$



Example 5.9. Fig.10 shows the graph of $nD_n^*(\{i\alpha_2\})$. α_2 has double large partial quotient a_2, a_3. In this case, η_1 is even and η_2 is odd. From Corollary 5.7, 40 hills of short period q_2 and one large hill of period q_3 are canceled with each other. We can see the shape of the graph like arches in Fig.10.

Remark 5.5. In the case of $\alpha = \log_{10} 7$, α has double large partial quotients a_7 and a_{62}. Because 7 is odd but 62 is even, hills of short period q_7 and one large hill of period q_{62} are canceled with each other. We can expect large arches like Fig.10. By the way, Setokuchi [22] claims that hills of period q_6 repeat more than 2.7×10^{27} times until $n < q_{61}$. From Corollary 5.7, we can expect that large arches will appear for $n > q_{62}$.

Fig.11 $\alpha_3 = [\,0; 2, 1, 40, 20, 1, 10, 2, 2, \dots\,]$.



Example 5.10. Fig.11 shows the graph of $nD_n^*(\{i\alpha_3\})$. α_3 has triple large partial quotients a_3, a_4 and a_6. In this case, note that $\eta_1(=3)$ is odd, $\eta_2(=4)$ and $\eta_3(=6)$ are even, so that $\eta_2+\eta_3(=10)$ is even. Thus, hills of period q_{η_2} are overlapped repeatedly on one large hill of period q_{η_3} and the height of overlapped hills top is $\frac{1}{4}a_{\eta_2}+\frac{1}{4}a_{\eta_3}(=7.5)$. Those overlapped hills and short hills of period q_{η_1} are canceled with each other and Fig.11 shows such compound arches. Note that $a_{\eta_1} > a_{\eta_2}+a_{\eta_3}$.

Remark 5.6. In the case of $\alpha = \log_{10} 37 - 1$, α has triple large partial quotients $a_{11}(=248)$, $a_{12}(=140)$ and $a_{14}(=85)$. In this case, $\eta_1(=11)$ is odd but $\eta_2(=12)$, $\eta_3(=14)$ are even. Note also that $a_{\eta_1} > a_{\eta_2}+a_{\eta_3}$ similarly as in Example 5.10. The numerical calculations is difficult in this case, but we can expect compound arches like Fig.11.

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