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Gale dual of the GKM graph with a complexity one axial function

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Abstract. The aim of this paper is to define a Gale dual of the GKM graph with a complexity one axial function, and show some basic properties. In particular, two properties which correspond to a pairwise linearly independence and a congruence relation of the axial function are proven.

Keywords: GKM graph; Gale dual.

1 Introduction

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be a configuration of vectors that span the whole $\mathbb{R}^n$. Form an $(m-n)\times m$ matrix $B = (b_{jk})$ whose rows form a basis in the space of linear relations between $\mathbf{a}_i$. The set of columns $\mathbf{b}_1, \ldots, \mathbf{b}_m (\in \mathbb{R}^{m-n})$ of B is called a *Gale dual* configuration of $\mathbf{a}_1, \ldots, \mathbf{a}_m$. One of the merits to consider such a configuration of vectors is to reduce the dimension of vectors. For example, the Gale dual of a configuration of $n+1$ vectors in $\mathbb{R}^n$ is nothing but the set of $n+1$ real numbers. The Gale dual provides an efficient tool for studying the combinatorics of higher dimensional polytopes with few vertices. In particular, the Gale dual (and the Gale diagram) is used to classify n-dimensional polytopes with $n+3$ vertices and to find interesting examples of the n-dimensional polytopes with $n+4$ vertices (see [BP]).

On the other hand, a GKM graph is a combinatorial counterpart of a GKM manifold. Guilleminn-Zara in [GZ] define this notion independently of the geometry. Roughly, a GKM graph is an m-valent graph whose oriented edges have labels on $\mathbb{Z}^n$. Therefore, one can regard each vertex of a GKM graph as a configuration of m vectors in $\mathbb{Z}^n$. This interpretation leads us to consider the ($\mathbb{Z}$-linear)Gale dual on each vertex of a GKM graph. In this paper, we study the basic properties of Gale duals of GKM graphs. In particular, Theorem 4.2 and Theorem 4.3 are the counterpart of the GKM conditions.

2 Gale dual of the labelled graph

In this section, we define a Gale dual of the labelled graph $\mathcal{G} = (\Gamma, \alpha)$.

2.1 Gale dual

We first introduce the Gale dual over $\mathbb{Z}$.

Let $\mathcal{A} := \{\mathbf{a}_1, \ \ldots, \ \mathbf{a}_m\}$ be the set of (possibly multiplicative) vectors in $\mathbb{Z}^n$ such that $\mathcal{A}$ spans $\mathbb{Z}^n$. In this paper, the number $m-n$ is called a *complexity* of $\mathcal{A}$. These vectors define the $(m\times n)$-matrix by

$$A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m \end{pmatrix}.$$

Then the Hermite normal form tells us that there is the matrix $U \in GL(n;\mathbb{Z})$ such that

$$H := UA = \underbrace{\left.\begin{pmatrix} 1 & * & \cdots & * & \cdots & * \\ 0 & 1 & \cdots & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & * \end{pmatrix}\right\} n}_{m} \tag{2.1}$$

The transposed $(n \times m)$-matrix A^T defines the $\mathbb{Z}$-liner map:

$$A^T : \mathbb{Z}^n \to \mathbb{Z}^m.$$

Using the Hermite normal form (2.1) again, we have that the linear map A^T is injective and there is no cotorsion, *i.e.*, its cokernel $\mathbb{Z}^m/\mathrm{im}(A^T)$ is isomorphic to $\mathbb{Z}^r$, where $r = m - n$. By applying [RT, Proposition 1.10], the $\mathbb{Z}$-basis of $\ker(A) \subset \mathbb{Z}^m$ is given by the last r rows of $V \in GL(m; \mathbb{Z})$ which gives the Hermite normal form $H' = VA^T$ for A^T. We also know that

$$\mathbb{Z}^m = \mathrm{im}(A^T) \oplus \ker(A).$$

Denote the surjective linear map induced from the following natural projection:

$$B : \mathbb{Z}^m \to \mathbb{Z}^m/\mathrm{im}(A^T) \simeq \ker(A) \simeq \mathbb{Z}^r.$$

By choosing the basis of $\mathbb{Z}^r (\simeq \ker(A))$ by the last r rows of $V \in GL(m; \mathbb{Z})$, the linear map B is represented by the $(m \times r)$-matrix

$$B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_m \end{pmatrix},$$

such that

$$BA^T = 0.$$

Moreover, the following sequence is the short exact sequence:

$$\mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^r.$$

Note that $\mathcal{A}^* := \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ spans $\mathbb{Z}^r$ because B is surjective. We call $\mathcal{A}^*$ *a Gale dual of* $\mathcal{A}$. In summary, we have the following definition.

Definition 2.1. Let $\mathcal{A} := \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be the set of (possibly multiplicative) vectors in $\mathbb{Z}^n$ such that $\mathcal{A}$ spans $\mathbb{Z}^n$. Put $(m \times n)$-matrix $A = (\mathbf{a}_1 \cdots \mathbf{a}_m)$. Assume that the $(m \times r)$-matrix B gives the following exact sequence:

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^r \longrightarrow 0.$$

Then the set of column vectors of B, say

$$\mathcal{A}^* := \{\mathbf{b}_1, \ldots, \mathbf{b}_m\} \subset \mathbb{Z}^r,$$

is called a *Gale dual* (configuration) of $\mathcal{A} \subset \mathbb{Z}^n$.

Remark 2.2. There are several choices of a Gale dual $\mathcal{A}^* \subset \mathbb{Z}^r$ of the given $\mathcal{A} \subset \mathbb{Z}^n$. In this paper, we call two Gale duals $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are *equivalent* if there is a matrix $X \in GL(r; \mathbb{Z})$ such that $X \cdot \mathcal{A}_1^* = \mathcal{A}_2^*$, *i.e.*, for $\mathcal{A}_1^* = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$, $\mathcal{A}_2^* = \{X\mathbf{b}_1, \ldots, X\mathbf{b}_m\} \subset \mathbb{Z}^r$. We denote two equivalent Gale duals by $\mathcal{A}_1^* \simeq \mathcal{A}_2^*$.

Remark 2.3. There is a definition of the $\mathbb{Z}$-linear Gale dual in [RT] which is more general than the definition as above. The definition as above is directly modified from the usual Gale dual over $\mathbb{R}$ (see [BP]).

We have the following fact:

Proposition 2.4. Let $\mathcal{A} \subset \mathbb{Z}^n$ be a set of m non-zero vectors which spans $\mathbb{Z}^n$ and $\mathcal{A}^* \subset \mathbb{Z}^r$ be its Gale dual, where $r = m - n$. Then we can take the Gale dual of $\mathcal{A}^*$ as $\mathcal{A}$, *i.e.*,

$$(\mathcal{A}^*)^* \simeq \mathcal{A}.$$

Proof. Let A be the matrix defined by $\mathcal{A}$ and B be the matrix defined by $\mathcal{A}^*$. The equation $BA^T = 0$ implies the equation $(BA^T)^T = AB^T = 0$. By using the Hermite normal form for B^T, we also have that B^T is injective. Moreover, by the previous argument, there is no torsion in $\mathrm{im}(A^T)$ and $\mathrm{im}(A^T) \simeq \mathbb{Z}^m/\ker(A) = \mathbb{Z}^m/\mathrm{im}(B^T)$. Therefore, there is no cotorsion for B^T. This establishes that the Gale dual of $\mathcal{A}^*$ is equivalent to $\mathcal{A}$. □

Example 2.5. Take the three vectors $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ in $\mathbb{Z}^2$ by

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then its Gale dual $\mathcal{A}^* = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ in $\mathbb{Z}$ is

$$\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

2.2 The canonical Gale dual of complexity one vector configurations

In this section, we consider the Gale dual of the case when the vector configuration $\mathcal{A}$ is a complexity one, *i.e.*, $\mathcal{A}=\{\mathbf{a}_1,\ \dots,\ \mathbf{a}_{n+1}\}\subset\mathbb{Z}^n$ and $\mathcal{A}$ spans $\mathbb{Z}^n$. We put the $n\times(n+1)$-matrix defined from $\mathcal{A}$ as follows:

$$A^T=\begin{pmatrix}\mathbf{a}_1^T\\ \mathbf{a}_2^T\\ \vdots\\ \mathbf{a}_{n+1}^T\end{pmatrix}=\begin{pmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{pmatrix}.$$

Denote the $(n\times n)$-matrix which removes the ith row vector from A^T as follows:

$$A_i^T=\begin{pmatrix}\mathbf{a}_1^T\\ \vdots\\ \mathbf{a}_{i-1}^T\\ \mathbf{a}_{i+1}^T\\ \vdots\\ \mathbf{a}_{n+1}^T\end{pmatrix}=\begin{pmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1(i-1)} & a_{2(i-1)} & \cdots & a_{n(i-1)}\\ a_{1(i+1)} & a_{2(i+1)} & \cdots & a_{n(i+1)}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{pmatrix}.$$

Then we have the following proposition:

Proposition 2.6. Let $\mathcal{A}=\{\mathbf{a}_1,\ \dots,\ \mathbf{a}_{n+1}\}\subset\mathbb{Z}^n$ be a complexity one vector configuration. Then a Gale dual of $\mathcal{A}$ is taken as the following set of integers:

$$\mathcal{A}^*=\{|A_1^T|,\ -|A_2^T|,\ |A_3^T|,\ \dots,(-1)^{n+1}|A_{n+1}^T|\}.$$

Proof. Let $A=(\mathbf{a}_1\ \cdots\ \mathbf{a}_{n+1})$ be the $n\times(n+1)$-matrix and

$$R=(|A_1^T|\ \ -|A_2^T|\ |A_3^T|\ \cdots\ (-1)^{n+1}|A_{n+1}^T|)$$

the $1\times(n+1)$-matrix. Then, we have

$$RA^T=|A_1^T|\mathbf{a}_1^T-|A_2^T|\mathbf{a}_2^T+|A_3^T|\mathbf{a}_3^T-\cdots+(-1)^{n+1}|A_{n+1}^T|\mathbf{a}_{n+1}^T.$$

It follows from the cofactor expansion that the 1st row (integer) of this matrix is as follows:

$$\begin{aligned}
&a_{11}|A_1^T|-a_{12}|A_2^T|+\cdots+(-1)^{n+1}a_{1(n+1)}|A_{n+1}^T|\\
=&a_{11}\begin{vmatrix}a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}-a_{12}\begin{vmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ a_{13} & a_{23} & \cdots & a_{n3}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}\\
&\qquad+\cdots+(-1)^{n+1}a_{1(n+1)}\begin{vmatrix}a_{11} & a_{21} & \cdots & a_{n1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{vmatrix}\\
=&\begin{vmatrix}a_{11} & a_{11} & a_{21} & \cdots & a_{n1}\\ 0 & a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}+\begin{vmatrix}0 & a_{11} & a_{22} & \cdots & a_{n2}\\ a_{12} & a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}\\
&\qquad+\cdots+\begin{vmatrix}0 & a_{11} & a_{22} & \cdots & a_{n2}\\ 0 & a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}\\
=&\begin{vmatrix}a_{11} & a_{11} & a_{21} & \cdots & a_{n1}\\ a_{12} & a_{12} & a_{22} & \cdots & a_{n2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{1(n+1)} & a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)}\end{vmatrix}=0.
\end{aligned}$$

Therefore, the 1st row of RA^T is zero. Similarly, we can prove that the other rows are zero. Hence, $RA^T = \mathbf{0}$. This implies that the composition map $\mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^{n+1} \xrightarrow{R} \mathbb{Z}$ is the zero map, *i.e.*, $\ker(R) \supset \mathrm{im}(A^T)$.

Next we assume that a non-zero element $\mathbf{x} = (x_1 \cdots x_{n+1})^T \in \mathbb{Z}^{n+1}$ satisfies that $R\mathbf{x} = 0$. By definition of R and applying the similar computations as above, we have

$$0 = R\mathbf{x} = x_1|A_1^T| - \cdots + (-1)^{n+1}x_{n+1}|A_{n+1}^T| = \begin{vmatrix} x_1 & a_{11} & a_{21} & \cdots & a_{n1} \\ x_2 & a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n+1} & a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)} \end{vmatrix}.$$

We put the final square matrix from this computation by $\left(\mathbf{x}\ A^T\right)$. This computation shows that $\mathrm{rk}\left(\mathbf{x}\ A^T\right) \leq n$. Therefore, the system of the equations

$$\left(\mathbf{x}\ A^T\right)\begin{pmatrix} l \\ \mathbf{y}' \end{pmatrix} = l\mathbf{x} + A^T\mathbf{y}' = \mathbf{0}$$

has the non-trivial solutions, where $l \in \mathbb{Z}$ and $\mathbf{y}' \in \mathbb{Z}^n$. Suppose that $(l\ \mathbf{y}')^T$ is a non-trivial solution. Since A^T is injective (because $\mathcal{A}$ spans $\mathbb{Z}^n$), if $l = 0$ then the equation $A^T\mathbf{y}' = -l\mathbf{x}$ implies that $\mathbf{y}' = \mathbf{0}$; however, this gives a contradiction to that $(l\ \mathbf{y}')^T$ is a non-trivial solution. Hence, we have $l\mathbf{x}(\neq \mathbf{0}) \in \mathrm{im}(A^T)$. Now because there is an i such that $|A_i^T| = \pm 1$, the cokernel $\mathbb{Z}^{n+1}/\mathrm{im}(A^T)$ has no-torsion. This implies that there exists an element $\mathbf{y} \in \mathbb{Z}^n$ such that

$$A^T\mathbf{y} = \mathbf{x}(\neq \mathbf{0}) \in \ker(R).$$

This shows that $\ker(R) \subset \mathrm{im}(A^T)$. Hence, we have $\ker(R) = \mathrm{im}(A^T)$ and the sequence $\mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^{n+1} \xrightarrow{R} \mathbb{Z}$ is exact.

We finally prove the sequence $\mathbb{Z}^n \xrightarrow{A^T} \mathbb{Z}^{n+1} \xrightarrow{R} \mathbb{Z}$ is the short exact sequence, *i.e.*, A^T is injective and R is surjective. Because the row vectors of A^T span $\mathbb{Z}^n$, the map $A^T : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ is injective. We shall show that the map $R : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is surjective. By using the Hermite normal form [RT, Proposition 1.10], there are a matrix

$$H = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$$

and the isomorphism $U : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ such that

$$UA^T = H.$$

Hence, we have

$$A^T = U^{-1}H.$$

Put $U^{-1}(\mathbf{e}_{n+1}) = \mathbf{a}$, where $\mathbf{e}_{n+1}$ is the $(n+1)$st vector of the standard basis of $\mathbb{Z}^{n+1}$. Consider the $(n+1) \times (n+1)$-matrix $(\mathbf{a}\ A^T)$. Then we have

$$\begin{aligned}(\mathbf{a}\ A^T) &= (\mathbf{a}\ U^{-1}H) \\ &= (U^{-1}\mathbf{e}_{n+1}\ U^{-1}H) \\ &= U^{-1}(\mathbf{e}_{n+1}\ H).\end{aligned}$$

Since $\det U^{-1} = \pm 1$ and $\det(\mathbf{e}_{n+1}\ H) = \pm 1$, we have

$$\det(\mathbf{a}\ A^T) = \pm 1.$$

By the definition of R, if we put $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{Z}^{n+1}$,

$$R\mathbf{a} = a_1|A_1^T| - a_2|A_2^T| + a_3|A_3^T| - \cdots + (-1)^{n+1}a_{n+1}|A_{n+1}^T| = \det(\mathbf{a}\ A^T) = \pm 1.$$

This establishes that R is surjective. □

2.3 Abstract graph

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an abstract, oriented graph with vertices $V(\Gamma)$ and edges $E(\Gamma)$. We assume that Γ is finite and connected, and there is no loops in $E(\Gamma)$ in this article. For an edge $e \in E(\Gamma)$, we use the following symbols:

- $i(e)$ the initial vertex of e;
- $t(e)$ the terminal vertex of e;
- $\overline{e} \in E(\Gamma)$ the orientation reversed edge of e.

Set

$$E_p(\Gamma) = \{e \in E(\Gamma) \mid i(e) = p\}.$$

An abstract graph Γ is called an *m-valent graph* if $|E_p(\Gamma)| = m$ for all $p \in V(\Gamma)$.

We say, two abstract graphs Γ and Γ' *are combinatorially equivalent*, if there is a bijective map

$$f : \Gamma = (V(\Gamma), E(\Gamma)) \to (V(\Gamma'), E(\Gamma')) = \Gamma'$$

such that $\overline{f(e)} = f(\overline{e})$ for all $e \in E(\Gamma)$ and the following diagram commutes (where the vertical map i in the diagram is the projection onto the initial vertex of an edge):

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{f} & E(\Gamma') \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ V(\Gamma) & \xrightarrow{f} & V(\Gamma'). \end{array}$$

2.4 Labeled graph

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an abstract m-valent graph. If there is a label $\alpha : E(\Gamma) \to \mathbb{Z}^n$ on edges of Γ, then we denote such labeled graph by $\mathcal{G} = (\Gamma, \alpha)$ and call it an *(m, n)-labeled graph*.

We say, two (m, n)-labeled graphs $\mathcal{G} = (\Gamma, \alpha)$ and $\mathcal{G}' = (\Gamma', \alpha')$ *are equivalent* (or *φ-equivalent*), denoted as $\mathcal{G} \cong \mathcal{G}'$, if there is a combinatorial equivalent map $f : \Gamma \to \Gamma'$ and an isomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that the following diagram is commutative:

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{\alpha} & \mathbb{Z}^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} \\ E(\Gamma') & \xrightarrow{\alpha'} & \mathbb{Z}^n. \end{array} \tag{2.2}$$

Set

$$\alpha_{(p)} = \{\alpha(e) \mid e \in E_p(\Gamma)\} \subset \mathbb{Z}^n.$$

The set of m vectors $\alpha_{(p)}$ gives the vector configuration in $\mathbb{Z}^n$ on each vertex. We assume that $\alpha_{(p)}$ spans $\mathbb{Z}^n$ for each $p \in V(\Gamma)$. Then we can define the Gale dual $(\alpha_{(p)})^*$ in $\mathbb{Z}^r$ $(r = m - n)$. Moreover, this gives a new label

$$\rho : E(\Gamma) \to \mathbb{Z}^r$$

such that $\rho_{(p)}$ is the Gale dual of $\alpha_{(p)}$. We denote (Γ, ρ) by $\mathcal{G}^*$ and call it a *Gale dual of the labeled graph* $\mathcal{G}$.

By Proposition 2.4, we have the following corollary:

Corollary 2.7. Let $\mathcal{G} = (\Gamma, \alpha)$ be an (m, n)-labeled graph and $\mathcal{G}^* = (\Gamma, \rho)$ be its Gale dual, *i.e.*, the $(m, m - n)$-labeled graph. Then there is the following isomorphism:

$$(\mathcal{G}^*)^* \cong \mathcal{G}.$$

3 Gale dual of the axial function of a GKM graph

In this section, we define the Gale dual of the axial function of a GKM graph. We first recall the complexity one GKM graph (see [K16]).

Let Γ be an m-valent graph. We define a label $\alpha : E(\Gamma) \to H^2(BT)$ on Γ. Recall that BT^n (often denoted by BT) is a classifying space of an n-dimensional torus T, and its cohomology ring (over $\mathbb{Z}$-coefficient) is isomorphic to the polynomial ring

$$H^*(BT) \simeq \mathbb{Z}[a_1, \ldots, a_n],$$

where a_i is a variable with $\deg a_i = 2$ for $i = 1, \ldots, n$. So its degree two part $H^2(BT)$ is isomorphic to $\mathbb{Z}^n$. Put a label by a function $\alpha : E(\Gamma) \to H^2(BT)$ on edges of Γ.

An *axial function* on Γ is a function $\alpha : E(\Gamma) \to H^2(BT^n)$ for $n \leq m$ which satisfies the following three conditions:

(1) $\alpha(e) = -\alpha(\overline{e})$;

(2) for each $p \in V(\Gamma)$, the set $\alpha_{(p)}$ is *pairwise linearly independent*, *i.e.*, each pair of elements in $\alpha_{(p)}$ are linearly independent in $H^2(BT)$;

(3) for all $e \in E(\Gamma)$, there exists a bijective map $\nabla_e : E_{i(e)}(\Gamma) \to E_{t(e)}(\Gamma)$ such that

1. $\nabla_{\overline{e}} = \nabla_e^{-1}$,
2. $\nabla_e(e) = \overline{e}$, and
3. for each $e' \in E_{i(e)}(\Gamma)$, the following relation (called a *congruence relation*) holds:

$$\alpha(\nabla_e(e')) - \alpha(e') \equiv 0 \mod \alpha(e) \in H^2(BT). \tag{3.1}$$

The collection $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a *connection* on the labelled graph (Γ, α); we denote the labelled graph with connection as (Γ, α, ∇). The conditions as above are called an *axiom of axial function.* In addition, we also assume the following condition:

(4) for each $p \in V(\Gamma)$, the set $\alpha_{(p)}$ spans $H^2(BT)$.

The axial function which satisfies (4) is called an *effective* axial function.

Definition 3.1. If an m-valent graph Γ is labeled by an axial function $\alpha : E(\Gamma) \to H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) *GKM graph*, and denoted by (Γ, α, ∇). If such α is effective, (Γ, α, ∇) is said to be an *(effective)* (m, n)*-type GKM graph.* In particular, we call an $(n+1, n)$-type GKM graph a *complexity one GKM graph.*

Then, the equivalence relation on GKM graphs can be defined as follows:

Definition 3.2. Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ and $\mathcal{G}' = (\Gamma', \alpha', \nabla')$ be GKM graphs. We say, $\mathcal{G}$ and $\mathcal{G}'$ *are equivalent* (or φ*-equivalent*), denoted by $\mathcal{G} \simeq \mathcal{G}'$, if there is a combinatorial equivalent map $f : \Gamma \to \Gamma'$ and an isomorphism $\varphi : H^2(BT^n) \to H^2(BT^n)$ such that the following diagram is commutative:

$$\begin{tikzcd} E(\Gamma) \arrow[r, "\alpha"] \arrow[d, "f"] & H^2(BT^n) \arrow[d, "\varphi"] \\ E(\Gamma') \arrow[r, "\alpha'"] & H^2(BT^n), \end{tikzcd} \tag{3.2}$$

i.e., labeled graphs (Γ, α) and (Γ', α') are equivalent; moreover, f preserves the connection, *i.e.*,

$$\begin{tikzcd} E_{i(e)}(\Gamma) \arrow[r, "\nabla_e"] \arrow[d, "f"] & E_{t(e)}(\Gamma) \arrow[d, "f"] \\ E_{i(f(e))}(\Gamma') \arrow[r, "\nabla'_{f(e)}"] & E_{t(f(e))}(\Gamma') \end{tikzcd} \tag{3.3}$$

is commutative for all $e \in E(\Gamma)$.

Now we may define the Gale dual of a GKM graph.

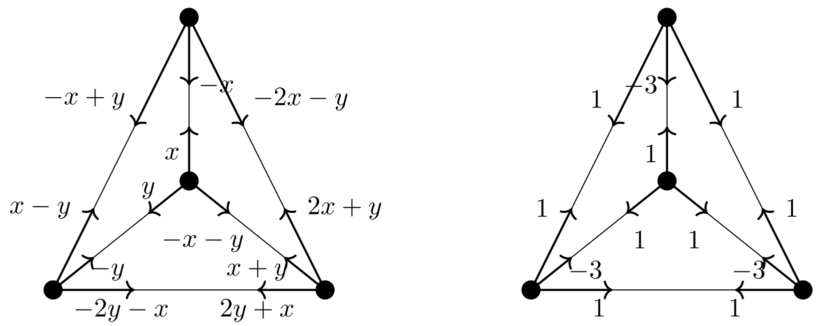

Figure 1: A complexity one GKM graph and its Gale dual, where $x,\ y$ are generators of $H^2(BT^2)$.

Definition 3.3 (Gale dual)**.** Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ be an (m, n)-type GKM graph. Take the Gale dual of (Γ, α), say (Γ, ρ), i.e., $(m, m-n)$-labeled graph. We call the pair with the connection $\mathcal{G}^* := (\Gamma, \rho, \nabla)$ a *Gale dual of a(n) (axial function of) GKM graph* $\mathcal{G}$.

By Corollary 2.7, we have the following corollary:

Corollary 3.4. Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ be an (m, n)-type GKM graph and $\mathcal{G}^* = (\Gamma, \rho, \nabla)$ be its Gale dual, *i.e.*, the $(m, m-n)$-labeled graph with the connection ∇. Then there is the following isomorphism:

$$(\mathcal{G}^*)^* \cong \mathcal{G}.$$

4 Two properties of a Gale dual of the complexity one GKM graph

In this section, we prove the main result of this paper. Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ be a complexity one GKM graph where Γ is an $(n+1)$-valent graph. Then, we may put the axial functions around a vertex $p \in V(\Gamma)$ as follows:

$$\alpha_{(p)} = \{\alpha(e_{1,p}), \ldots, \alpha(e_{n+1,p})\} \subset H^2(BT^n) \simeq \mathbb{Z}^n.$$

Since the complexity one GKM graph satisfies the effectiveness condition, we may assume that $\alpha_{(p)}$ spans $H^2(BT^n) \simeq \mathbb{Z}^n$. Then we can define its Gale dual $\rho_{(p)}$ as follows:

$$\rho_{(p)} = \{\rho(e_{1,p}), \ldots, \rho(e_{n+1,p})\} \subset \mathbb{Z}$$

such that $(\rho(e_{1,p}), \ldots, \rho(e_{n+1,p})) \in \mathbb{Z}^{n+1}$ is a primitive vector and

$$\sum_{i=1}^{n+1} \rho(e_{i,p})\alpha(e_{i,p}) = 0.$$

We define the $n \times (n+1)$-matrix

$$A_{(p)} = \begin{pmatrix} \alpha(e_{1,p}) \\ \vdots \\ \alpha(e_{n+1,p}) \end{pmatrix}$$

We first claim the following lemma:

Lemma 4.1. *The Gale dual $\rho_{(p)}$ can be taken as*

$$\rho(e_{i,p}) = (-1)^{i+1}|A_{(p)}(i)|$$

where $A_{(p)}(i)$, $i = 1, \ldots, n+1$, is the $n \times n$-square matrix which removes the ith row from $A_{(p)}$.

Proof. Because of Proposition 2.6, this statement is straightforward. □

Now we may prove the two properties of Gale duals of GKM graphs. The following first theorem corresponds to the pairwise linearly independentness of GKM graphs.

Theorem 4.2. *For every $p \in V(\Gamma)$, there are mutually distinct $s, t, u \in [n+1] := \{1, 2, \ldots, n+1\}$ such that*

$$\rho(e_{s,p})\rho(e_{t,p})\rho(e_{u,p}) \neq 0.$$

Proof. Let $\rho_{(p)}$ be the Gale dual of the complexity one GKM graph (Γ, α, ∇). By definition, we have

$$\sum_{i=1}^{n+1} \rho(e_{i,p})\alpha(e_{i,p}) = \mathbf{0}, \tag{4.1}$$

for all $p \in V(\Gamma)$.

Since (Γ, α, ∇) is a complexity one, for some $s \in [n+1]$, there exists $r_k \in \mathbb{Z}$ ($k \in [n+1] \setminus \{s\}$) such that

$$\alpha(e_{s,p}) = \sum_{k \neq s} r_k \alpha(e_{k,p}). \tag{4.2}$$

Substituting (4.2) to (4.1), we have

$$\sum_{k \neq s} \{\rho(e_{k,p}) + r_k \rho(e_{s,p})\} \alpha(e_{k,p}) = \mathbf{0}.$$

Because of the effectiveness condition, $\{\alpha(e_{k,p}) \mid k \in [n+1] \setminus \{s\}\}$ spans $H^2(BT^n)$. Thus, we have

$$\rho(e_{k,p}) = -r_k \rho(e_{s,p}) \tag{4.3}$$

for all $k \neq s$. If we assume $\rho(e_{s,p}) = 0$, then $\rho(e_{k,p}) = 0$ for all $k \in [n+1] \setminus \{s\}$ by (4.3). However, this gives a contradiction to that $(\rho(e_{1,p}), \ldots, \rho(e_{n+1,p})) \in \mathbb{Z}^{n+1}$ is a primitive vector. Therefore, we have $\rho(e_{s,p}) \neq 0$.

Moreover, by (4.2) again, if only one r_k is non-zero and the others are zero, then $\alpha(e_{s,p}) = r_k \alpha(e_{k,p})$. This contradicts to the pairwise linearly independence of the axial function. Therefore, at least two distinct integers, say r_t and r_u ($t, u \in [n+1] \setminus \{s\}$), must be non-zero. Together with (4.3), we have that $\rho(e_{t,p}),\ \rho(e_{u,p}) \neq 0$. This establishes the statement. □

The next theorem is the 2nd main result of this paper. This property corresponds to the congruence relation and $\alpha(e) = -\alpha(\overline{e})$.

Theorem 4.3. *Let $\mathcal{G}$ be a complexity one GKM graph and $\mathcal{G}^*$ its Gale dual. Let $p \in V(\Gamma)$ and $E_p(\Gamma) = \{e_1, \ldots, e_{n+1}\}$. Fix $e \in E_p(\Gamma)$. Then, for every $e_j \neq e$, the following equation holds:*

$$|\rho(\nabla_e(e_j))| = |\rho(e_j)|.$$

Proof. For $e = e_{i,p} \in E(\Gamma)$, we put $i(e) = p$, $t(e) = q$ and

$$A_{(p)} = \begin{pmatrix} \alpha(e_{1,p}) \\ \vdots \\ \alpha(e_{n+1,p}) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)} \end{pmatrix},$$

$$A_{(q)} = \begin{pmatrix} \alpha(\nabla_e(e_{1,p})) \\ \vdots \\ \alpha(\nabla_e(e_{n+1,p})) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \\ b_{1(n+1)} & b_{2(n+1)} & \cdots & b_{n(n+1)} \end{pmatrix}.$$

By the congruence relations (3.1), there are $k_1, \ldots, k_{n+1} \in \mathbb{Z}$ such that

$$A_{(q)} - A_{(p)} = \begin{pmatrix} \alpha(\nabla_e(e_{1,p})) - \alpha(e_{1,p}) \\ \vdots \\ \alpha(\nabla_e(e_{n+1,p})) - \alpha(e_{n+1,p}) \end{pmatrix} = \begin{pmatrix} \alpha(e)k_1 \\ \vdots \\ \alpha(e)k_{n+1} \end{pmatrix}. \tag{4.4}$$

Take $j \neq i$. Then, by Lemma 4.1 and (4.4), we have

$$|\rho(e_{j,p})| = |\det A_{(p)}(j)| = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1(j-1)} & a_{2(j-1)} & \cdots & a_{n(j-1)} \\ a_{1(j+1)} & a_{2(j+1)} & \cdots & a_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1(n+1)} & a_{2(n+1)} & \cdots & a_{n(n+1)} \end{vmatrix}$$

$$= \begin{vmatrix} b_{11} - a_{1i}k_1 & b_{21} - a_{2i}k_1 & \cdots & b_{n1} - a_{ni}k_1 \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} - a_{1i}k_{j-1} & b_{2(j-1)} - a_{2i}k_{j-1} & \cdots & b_{n(j-1)} - a_{ni}k_{j-1} \\ b_{1(j+1)} - a_{1i}k_{j+1} & b_{2(j+1)} - a_{2i}k_{j+1} & \cdots & b_{n(j+1)} - a_{ni}k_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(n+1)} - a_{1i}k_{n+1} & b_{2(n+1)} - a_{2i}k_{n+1} & \cdots & b_{n(n+1)} - a_{ni}k_{n+1} \end{vmatrix}.$$

Therefore, by using the basic property of the determinant,

$$|\rho(e_{j,p})| = \begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} & b_{2(j-1)} & \cdots & b_{n(j-1)} \\ b_{1(j+1)} & b_{2(j+1)} & \cdots & b_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(n+1)} & b_{2(n+1)} & \cdots & b_{n(n+1)} \end{vmatrix} \tag{4.5}$$

$$- k_1 \begin{vmatrix} a_{1i} & a_{2i} & \cdots & a_{ni} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} & b_{2(j-1)} & \cdots & b_{n(j-1)} \\ b_{1(j+1)} & b_{2(j+1)} & \cdots & b_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(n+1)} & b_{2(n+1)} & \cdots & b_{n(n+1)} \end{vmatrix} - k_2 \begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ a_{1i} & a_{2i} & \cdots & a_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} & b_{2(j-1)} & \cdots & b_{n(j-1)} \\ b_{1(j+1)} & b_{2(j+1)} & \cdots & b_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(n+1)} & b_{2(n+1)} & \cdots & b_{n(n+1)} \end{vmatrix}$$

$$\cdots - k_{n+1} \begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} & b_{2(j-1)} & \cdots & b_{n(j-1)} \\ b_{1(j+1)} & b_{2(j+1)} & \cdots & b_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \\ a_{1i} & a_{2i} & \cdots & a_{ni} \end{vmatrix}.$$

Because $\alpha(e) = -\alpha(\overline{e})$, we have that $\alpha(e_{i,p}) = -\alpha(\nabla_e(e_{i,p}))$; therefore, by the definition of $A_{(q)}$ as above (this gives a choice of the order of $E_q(\Gamma)$), we have that

$$(a_{1i} \ \cdots \ a_{ni}) = -(b_{1i} \ \cdots \ b_{ni}).$$

Hence, by $j \neq i$, the equation (4.5) and Lemma 4.1 give that

$$|\rho(e_{j,p})| = \begin{vmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(j-1)} & b_{2(j-1)} & \cdots & b_{n(j-1)} \\ b_{1(j+1)} & b_{2(j+1)} & \cdots & b_{n(j+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1(n+1)} & b_{2(n+1)} & \cdots & b_{n(n+1)} \end{vmatrix} = |\det A_{(q)}(j)| = |\rho(\nabla_e(e_{j,q}))|.$$

This establishes the statement. $\square$

Remark 4.4. The integers $k_1, \ldots, k_{n+1}$ appeared in the proof are nothing but $c_{(\Gamma,\alpha,\nabla)}$ in [K19].

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References

[BP] V. M. Buchstaber and T. E. Panov, Toric Topology, Math. Surveys and Monographs, vol.204, Amer. Math. Soc., Providence, RI, 2015.

[GZ] V. Guillemin and C. Zara, *One-skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J. **107**, 2 (2001), 283–349.

[K16] S. Kuroki, *Classification of effective GKM graphs with combinatorial type K_4*, Bull. of the Okayama U. of Science, **52** A (2016), 7–15.

[K19] S. Kuroki, *Upper bounds for the dimension of tori acting on GKM manifolds*, J. of the Math. Soc. of Japan, **71**, No. 2 (2019), 483–513.

[RT] M. Rossi and L. Terracini, *$\mathbb{Z}$-linear Gale duality and poly weighted spaces (PWS)*, arXiv:1501.05244.