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Secant type approach via approximation of logarithmic derivative

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A numerical method for solving a nonlinear equation with multiple roots, iteratively, is proposed. The proposed method is a kind of secant method, namely a derivative free method. It is shown that the convergence of the present method. Moreover, it is shown the efficiency of the present method by numerical experiments. Moreover, we apply the approximation to formulate two second order derivative free methods.

Keywords: Secant method, nonlinear equation, logarithmic derivative, second derivative free method.

1. Introduction

Let us consider an iterative solution for the nonlinear equation with the multiple roots such as

$$f(x) = 0 \tag{1}$$

where $f(x)$ is a real function that is convex around the neighbor of the solution a. A popular approach to seek the numerical solution is the Newton iterative method expressed as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{2}$$

in which the convergence rate of (2) is linear because of the multiplicity of the root a[1]. If we have the information about the multiplicity of the root m, we have the second order convergence iterative method[1] such as

$$x_{k+1} = x_k - m\frac{f(x_k)}{f'(x_k)}\ . \tag{3}$$

For a given (1), sometime, it is difficult to determine the multiplicity of the root m. The secant method for (1) is expressed as

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})f(x_k)}{f(x_k) - f(x_{k-1})} \tag{4}$$

in which the convergence rate of the method is super-linear for simple roots. If the equation (1) has a multiple solution, the secant method has the first order convergence[2),3)]. As increasing the multiplicity, the rate of convergence becomes slow.

In this paper, we resent a secant type method in which the method is of first order. However the method efficient is same or better than the case of applying the Newton method (2). The convergence theorem for the present method is proved. By some numerical experiments, it is shown that the present method is the effective derivative free iterative method without the parameter related to the multiplicity for the equation with multiple roots. Moreover, we apply the approximation to formulate two second order derivative free methods.

2. Secant type iterative method and convergence

The Newton iteration (2) is rewritten into

$$x_{k+1} = x_k - \frac{1}{[\log f(x_k)]'} \tag{5}$$

with the logarithmic derivative (LD). By applying the approximation of LD such as

$$[\log f(x_k)]' \approx \frac{\log f(x_k) - \log f(x_{k-1})}{x_k - x_{k-1}} \tag{6}$$

to (5) we obtain

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\log f(x_k) - \log f(x_{k-1})} \quad . \tag{7}$$

The iteration (7) had to evaluate the logarithmic function, then $f(x_k)$ must be positive. However, we reform (7) to

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{\log\left(\frac{f(x_k)}{f(x_{k-1})}\right)} \tag{8}$$

where the restriction in this case is of some sign for $f(x_k)$ and $f(x_{k-1})$. Without positivity for $f(x_k)$ and $f(x_{k-1})$, the iteration as follows:

$$x_{k+1} = x_k - \frac{2(x_k - x_{k-1})}{\log\left(\left(\frac{f(x_k)}{f(x_{k-1})}\right)^2\right)} \tag{9}$$

is available. For the secant method, we have the following theorem:

Theorem 1: *Assume $f(x)$ is a real function that is convex around the neighbor of the solution a. Without loss of generality, we also assume that $a < x_1 < x_0$ and $0 < f(x_1) < f(x_0)$. Then the sequence generated by (4) is the monotone decreasing converging to the solution a.*

Proof: The sign of tangent of the secant in this case is positive. Therefor, $x_2 < x_1 < x_0$. In general, $x_{k+1} < x_k < x_{k-1}$. From the convexity around the neighbor of the solution a, $a < x_k$ for any k. Then we have the theorem.

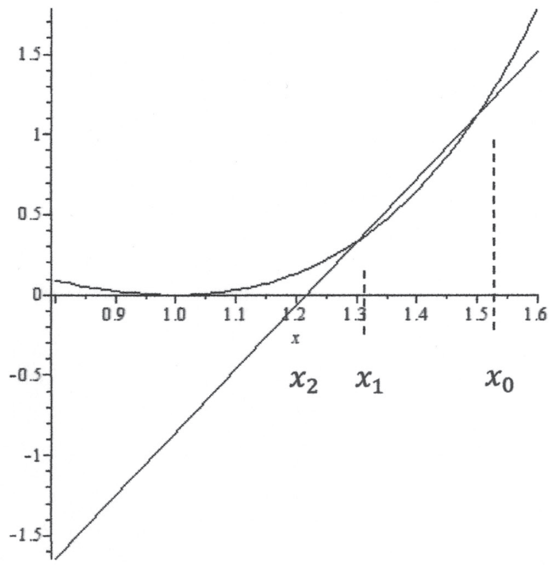

Fig 1: This figure illustrates the situation in the theorem 1.

Next, we discuss the convergence of the iteration (9).

Theorem 2: *Suppose same assumption in theorem 1 and the secant method is convergent for the equation* (1). *Then the proposed iteration* (9) *converges to same fixed point by the secant method.*

Proof : From (7) and (8), equivalently (9), we obtain

$$x_{k+1} = x_k - \frac{f(x_k)-f(x_{k-1})}{\log f(x_k)-\log f(x_{k-1})}\frac{x_k-x_{k-1}}{f(x_k)-f(x_{k-1})} \tag{10}$$

where

$$\frac{f(x_k)-f(x_{k-1})}{\log f(x_k)-\log f(x_{k-1})}$$

is the logarithmic mean of $f(x_k)$ and $f(x_{k-1})$. Therefor,

$$f(x_k) \le \frac{f(x_k)-f(x_{k-1})}{\log f(x_k)-\log f(x_{k-1})} \le f(x_{k-1}) \ . \tag{11}$$

According to the theorem 1 and the convergence of the secant method, we have the convergence of the proposed method by (11).

3. Application to the Chebyshev and the Halley iterations

We discuss to apply the proposed secant method to Chebyshev and the Halley methods. Those iterative methods have the term such as

$$L_f = \frac{f(x)f''(x)}{f'(x)^2} \ . \tag{12}$$

With (12), the Chebyshev and the Halley iterations are expressed as

$$\text{(Chebyshev method)} \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left(1+\frac{1}{2}L_f\right),$$

$$\text{(Halley method)} \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left(1-\frac{1}{2}L_f\right)^{-1}$$

respectively. Since

$$\frac{f''(x_k)}{f'(x_k)} = [\log f'(x_k)]' \approx \frac{1}{x_k-x_{k-1}}\log\frac{f'(x_k)}{f'(x_{k-1})} \ ,$$

we obtain iterative methods :

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left(1+\frac{1}{2}l_f\right) \tag{13}$$

and

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}\left(1-\frac{1}{2}l_f\right)^{-1} \ , \tag{14}$$

respectively, in which

$$l_f = \left(\log\frac{f'(x_k)}{f'(x_{k-1})}\right)\Big/\left(\log\frac{f(x_k)}{f(x_{k-1})}\right) \ . \tag{15}$$

4. Numerical experiments

In order to discuss the numerical efficiency of the proposed method, it is shown the numerical results compared with the Newton method and the secant method. Set the equation :

$$(x-1)^m \exp(ax) = 0 \tag{16}$$

where the multiplicity m and the positive real parameter a are given. Numerical solutions are evaluated by Maple 12 with the decimal 120 digits, the convergence criteria: $|x_k - 1| < 10^{-15}$, initial values $x_0 = 1.5,\ x_1 = 1.49$ for the secant method and the present method. For the Newton iteration, we use the initial value $x_0 = 1.5$. Table 1 is the results of the number of iterations for $a = 1$. The computational order of convergence (COC) defined as

$$\mathrm{COC} = \frac{\log\left(\left|\frac{x_{k+1}-1}{x_k-1}\right|\right)}{\log\left(\left|\frac{x_k-1}{x_{k-1}-1}\right|\right)}$$

for the problem. In the case of the present method, $\mathrm{COC} \approx 1$ which is same as the Newton method.

Table 1: Numerical resuts (the number of itertions) for (12). NC: no conergence.

Method	m=2	m=3	m=4	m=5	m=6	m=7	m=8
Newton	50	85	119	153	187	221	235
Secant	71	121	172	220	269	318	NC
(9)	55	57	96	132	167	201	233
(13)	35	58	81	104	127	149	172
(14)	31	50	67	84	101	111	171

5. Conclusion

The proposed method (9) is a derivative free iterative method as the secant method. The number of iteration is approximately equal to the case of the Newton method. As the multiplicity of the root is increased, the ratio of the number of iterations (NI): R[P/NR]=(NI of the present method (9))/(NI of the Newton-Raphson method (2)) become smaller. The present method converges to the solution which is a multiple root when the secant method is convergence. By the numerical experiments, the convergence rate of the present method (9) is of first order. Moreover, we apply the approximation for the logarithmic derivative to the method with the second order derivative as the Chebyshev iteration and the Halley iteration. We proposed the iterations (13) and (14) which are a kind of second derivative free methods.

Refernces

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