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Classification of effective GKM graphs with combinatorial type K_4

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Abstract. In this article, we classify GKM graphs (Γ, α, ∇) with effective axial function, where the graph Γ is combinatorially equivalent to the complete graph K_4. As a result, they are equivalent to one of the following distinct GKM graphs: $\mathcal{G}_{st}$; $\mathcal{G}_0(1,1)$; $\mathcal{G}_0(-1,-1)$; $\mathcal{G}_0(k,\pm1)$; $\mathcal{G}_1(m)$; $\mathcal{G}_2$; $\mathcal{G}_3$, where k and m are integers which satisfy $|k| \geq 2$ and $m \geq 1$.

Keywords: GKM graph; torus action; equivariant cohomology.

1 Introduction

The toric geometry may be regarded as the geometry of the spaces with complexity zero torus actions, i.e., the real $2n$-dimensional (complex n-dimensional) with nice T^n-actions. As is well-known, the theory of toric geometry is successfully developed and it is widely studied in several different areas of mathematics: algebraic geometry, symplectic geometry or topology, etc (see e.g. [BP, Od]). Recently, the theory about manifolds with complexity one torus actions (i.e., $(2n+2)$-dimensional manifolds with n-dimensional torus actions) are also studied, in particular, in algebraic geometry and symplectic geometry (see e.g. [KT1, KT2, Ti]). However, unlike toric geometry, from topological point of view, the spaces with complexity one torus actions is still developing (also see [Ku1]). In order to develop such theory, it may be useful to have good examples.

In this article, we introduce some examples from "combinatorial" point of view by classifying the GKM graphs (introduced in [GZ]) with combinatorial type K_4, i.e., the complete graph with four vertices. Recall that the GKM graphs are the combinatorial counterpart of the GKM manifolds which contains wide classes of manifolds with torus actions. For example, by classification of toric manifolds, a toric manifold whose GKM graph is K_4 is nothing but the 3-dimensional complex projective space $\mathbb{C}P^3$ with the standard torus action (also see Figure 1). Due to the definition of GKM graph, other GKM graphs whose combinatorial type is K_4 must be induced from the complexity one GKM manifolds (if there are geometric counterparts). Therefore, to classify such GKM graphs should be useful to study the complexity one GKM manifolds in the future.

The goal of this article is to classify GKM graphs with combinatorial type K_4 up to equivalence, and the main theorem can be stated as follows (see propositions in Section 3):

Theorem 1.1. *The GKM graph (K_4, α, ∇) is equivalent to one of the following GKM graphs:*

$$\mathcal{G}_{st},\ \mathcal{G}_0(1,1),\ \mathcal{G}_0(-1,-1),\ \mathcal{G}_0(k,\pm1),\ \mathcal{G}_1(m),\ \mathcal{G}_2,\ \mathcal{G}_3,$$

where k and m are integers such that $|k| \geq 2$ and $m \geq 1$.

The organization of this paper is as follows. In Section 2, we recall the basics of GKM graphs. In Section 3, we prove Theorem 1.1. In the final section (Section 4), we propose some questions derived from our classification.

2 GKM graph

2.1 Definition

We first recall the definition of GKM graph. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an *abstract graph* with vertices $V(\Gamma)$ and oriented edges $E(\Gamma)$. For the given orientation on $e \in E(\Gamma)$, we denote its initial vertex by $i(e)$ and its terminal vertex by $t(e)$. The symbol $\overline{e} \in E(\Gamma)$ represents the edge e with its orientation reversed. We assume no loops in $E(\Gamma)$ and Γ is connected in this article. Set

$$E_p(\Gamma) = \{e \in E(\Gamma) \mid i(e) = p\}.$$

A finite connected graph Γ is called an *m-valent graph* if $|E_p(\Gamma)| = m$ for all $p \in V(\Gamma)$.

Let Γ be an m-valent graph. We next define a label $\alpha : E(\Gamma) \to H^2(BT)$ on Γ. Recall that BT^n (often denoted by BT) is a classifying space of an n-dimensional torus T, and its cohomology ring (over $\mathbb{Z}$-coefficient) is isomorphic to the polynomial ring

$$H^*(BT) \simeq \mathbb{Z}[a_1, \ldots, a_n],$$

where a_i is a variable with $\deg a_i = 2$ for $i = 1, \ldots, n$. So its degree 2 part $H^2(BT)$ is isomorphic to $\mathbb{Z}^n$. Put a label by a function $\alpha : E(\Gamma) \to H^2(BT)$ on edges of Γ. Set

$$\alpha_{(p)} = \{\alpha(e) \mid e \in E_p(\Gamma)\} \subset H^2(BT).$$

An *axial function* on Γ is the function $\alpha : E(\Gamma) \to H^2(BT^n)$ for $n \leq m$ which satisfies the following three conditions:

(1) $\alpha(e) = -\alpha(\overline{e})$;

(2) for each $p \in V(\Gamma)$, the set $\alpha_{(p)}$ is *pairwise linearly independent*, i.e., each pair of elements in $\alpha_{(p)}$ is linearly independent in $H^2(BT)$;

(3) for all $e \in E(\Gamma)$, there exists a bijective map $\nabla_e : E_{i(e)}(\Gamma) \to E_{t(e)}(\Gamma)$ such that

1. $\nabla_{\overline{e}} = \nabla_e^{-1}$,
2. $\nabla_e(e) = \overline{e}$, and
3. for each $e' \in E_{i(e)}(\Gamma)$, the following relation (called a *congruence relation*) holds:

$$\alpha(\nabla_e(e')) - \alpha(e') \equiv 0 \mod \alpha(e) \in H^2(BT). \tag{2.1}$$

The collection $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a *connection* on the labelled graph (Γ, α); we denote the labelled graph with connection as (Γ, α, ∇). The conditions as above are called an *axiom of axial function*. In addition, we also assume the following condition:

(4) for each $p \in V(\Gamma)$, the set $\alpha_{(p)}$ spans $H^2(BT)$.

The axial function which satisfies (4) is called an *effective* axial function.

Definition 2.1 (GKM graph [GZ]). If an m-valent graph Γ is labeled by an axial function $\alpha : E(\Gamma) \to H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) *GKM graph*, and denoted as (Γ, α, ∇). If such α is effective, (Γ, α, ∇) is said to be an *(effective) (m, n)-type GKM graph.*

In this article, we only consider a complete graph with four vertices, denoted as K_4, i.e., the 3-valent graph with four vertices. An effective GKM graph whose combinatorial structure is K_4 must be a $(3, 3)$-type or $(3, 2)$-type GKM graph (see Figure 1).

2.2 Equivalence relation

We next define the equivalence relation among GKM graphs. We call two abstract graphs Γ and Γ' are *combinatorially equivalent* if there is a bijective map

$$f : \Gamma = (V(\Gamma), E(\Gamma)) \to (V(\Gamma'), E(\Gamma')) = \Gamma'$$

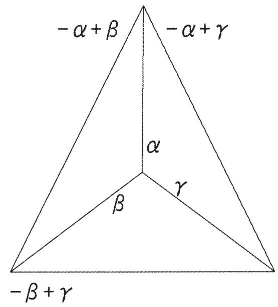

Figure 1: A (3,3)-type GKM graph $\mathcal{G}_{st}$. We omit the label on the different direction because it is determined automatically by the axiom (1).

such that $\overline{f(e)} = f(\overline{e})$ for all $e \in E(\Gamma)$ and the following diagram commutes:

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{f} & E(\Gamma') \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ V(\Gamma) & \xrightarrow{f} & V(\Gamma') \end{array}$$

where i is the projection onto the initial vertex of an edge. Then, the equivalence relation on GKM graphs can be defined as follows:

Definition 2.2. Let $\mathcal{G} = (\Gamma, \alpha, \nabla)$ and $\mathcal{G}' = (\Gamma', \alpha', \nabla')$ be GKM graphs. We call $\mathcal{G}$ and $\mathcal{G}'$ are *equivalent* (or *ρ-equivalent*), denoted as $\mathcal{G} \simeq \mathcal{G}'$, if there is a combinatorial equivalent map $f : \Gamma \to \Gamma'$ and an isomorphism $\rho : H^2(BT^n) \to H^2(BT^n)$ such that the following diagrams are commutative:

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{\alpha} & H^2(BT^n) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \rho} \\ E(\Gamma') & \xrightarrow{\alpha'} & H^2(BT^n) \end{array} \tag{2.2}$$

and

$$\begin{array}{ccc} E_{i(e)}(\Gamma) & \xrightarrow{\nabla_e} & E_{t(e)}(\Gamma) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ E_{i(f(e))}(\Gamma') & \xrightarrow{\nabla'_{f(e)}} & E_{t(f(e))}(\Gamma') \end{array} \tag{2.3}$$

for all $e \in E(\Gamma)$.

3 The classification

In this section, we classify GKM graphs with combinatorial type K_4.

3.1 Classification of abstract connections

We first ignore the axial functions and classify the possible connections on K_4 up to equivalence. Namely we classify the following *abstract connection* on K_4:

$$\nabla = \{\nabla_e \mid e \in E(K_4),\ \nabla_{\overline{e}} = \nabla_e^{-1},\ \nabla_e(e) = \overline{e}\}.$$

We denote K_4 with an abstract connection ∇ as (K_4, ∇). We define two (K_4, ∇) and (K_4, ∇') are *equivalent* if there is a combinatorial equivalent map $f : K_4 \to K_4$ such that f satisfies the commutative diagram (2.3). The following result is the classification of such (K_4, ∇).

Lemma 3.1. *There are exactly seven abstract connections, say ∇^{st} and $\nabla^{(k)}$ ($k = 1, \ldots, 6$), on K_4 up to equivalence.*

Proof. It is easy to check that there is the unique connection on $\mathcal{G}_{st}$ in Figure 1. We call this connection the standard connection ∇^{st}.

In order to get the other abstract connections, we may change the bijections ∇_e^{st} (and $\nabla_{\overline{e}}^{st}$) for $e \in E(K_4)$. By definition $\nabla_{\overline{e}} = \nabla_e^{-1}$, we may only consider the one direction of $e \in E(K_4)$, i.e., we may only think bijections on 6 edges. Moreover, by using definition $\nabla_e(e) = \overline{e}$ and the fact that K_4 is a 3-valent graph, there are two possible bijections on each edge $e \in E(K_4)$.

We first consider the case when only one bijection ∇_e^{st} on an edge $e \in E(K_4)$ is changed, denote such connection as $\nabla^{1,e}$. Because K_4 is the complete graph, for any other edge $e' \in E(K_4)$ two $(K_4, \nabla^{1,e})$ and $(K_4, \nabla^{1,e'})$ are equivalent. Therefore, the abstract connection which satisfies that the only one bijection is different from ∇^{st} is unique up to equivalence; therefore, we can denote it as $\nabla^{(1)}$.

Because K_4 is the complete graph, we can apply the similar method for the other cases when bijections on k edges in the connection are different from the standard connection. Namely, there is the unique connection $\nabla^{(k)}$ (up to equivalence) whose exactly k bijections $\nabla_e^{(k)}$ are different from those in the standard connection ∇^{st}, where $k = 1, \ldots, 6$. This establishes the statement. □

3.2 Classification of all axial functions

Now we may prove the main theorem. In the beginning, we introduce the following well-known fact from the toric geometry:

Proposition 3.2. The $(3,3)$-type GKM graph (K_4, α, ∇) is equivalent to $\mathcal{G}_{st}$ (Figure 1).

Therefore, we may only consider the case when (K_4, α, ∇) is a $(3,2)$-type GKM graph. We first show a rough classification.

Lemma 3.3. *The $(3,2)$-type GKM graph $(K_4, \alpha, \nabla^{st})$ is equivalent to $\mathcal{G}_0(m,n)$ (Figure 2), where m, n are non-zero integers which satisfy one of the following conditions: $m = \pm 1$, $n = \pm 1$, $n = -m$ or $n = 2 - m$.*

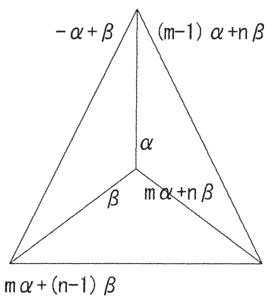

Figure 2: A (3,2)-type GKM graph $\mathcal{G}_0(m,n)$ with the standard connection ∇^{st}, where m and n are non-zero integers.

Proof. To show the statement, we use the figures, see Figure 3 and Figure 4.

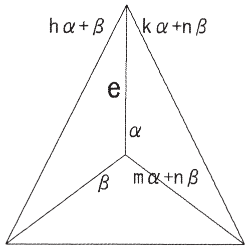

Figure 3: The axial functions around e.

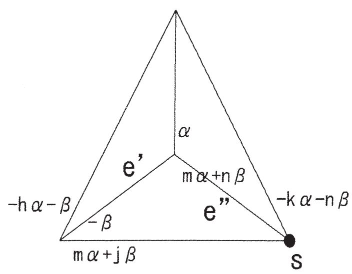

Figure 4: The axial functions around e', e''.

Fix the axial function around the middle vertex like in the Figure 3, i.e., α, β and $m\alpha+n\beta$ for the fixed basis $\alpha, \beta \in H^2(BT^2)$. Because they are pairwise linearly independent, we may choose $m, n \in \mathbb{Z} \setminus \{0\}$. Recall that the connection is the standard connection ∇^{st}. So by using the congruence relation along $e \in E(K_4)$, we also have integers $k, h \in \mathbb{Z}$ in Figure 3. Next, by using the congruence relation along $e', e'' \in E(K_4)$ (see Figure 4), we have $h = -1$, $k = m - 1$. By routine work, we also have $j = n - 1$.

In our article, we also assume that the effectiveness condition, i.e., the condition (4) in Section 2.1. Therefore, to satisfy the condition (4) around the vertex s (see Figure 4), we have one of the following equations:

$$m = \pm 1, \ n = \pm 1, \ \text{or}, \ m + n - 1 = \pm 1.$$

This establish the statement. □

Proposition 3.4. The GKM graphs which appeared in Lemma 3.3 are equivalent to one of the following distinct GKM graphs:

$$\mathcal{G}_0(1, 1), \quad \mathcal{G}_0(-1, -1), \quad \mathcal{G}_0(k, 1), \quad \mathcal{G}_0(k, -1)$$

where $|k| \geq 2$.

Proof. It is easy to check that $\mathcal{G}_0(m, n) \simeq \mathcal{G}_0(n, m)$ for all $m, n \in \mathbb{Z}\setminus\{0\}$. Therefore, we may write $\mathcal{G}_0(m, n)$ as $\mathcal{G}_0(k, 1)(\simeq \mathcal{G}_0(1, k))$, $\mathcal{G}_0(k, -1)(\simeq \mathcal{G}_0(-1, k))$, $\mathcal{G}_0(h, -h)(\simeq \mathcal{G}_0(-h, h))$ or $\mathcal{G}_0(j, 2-j)(\simeq \mathcal{G}_0(2-j, j))$ for some integers k, h, j such that $k \neq 0$, $h \geq 2$, $j \geq 4$.

We first consider $\mathcal{G}_0(k, 1)$ and $\mathcal{G}_0(-k, k)$ as Figure 5. Then, the identity map on two graphs with the

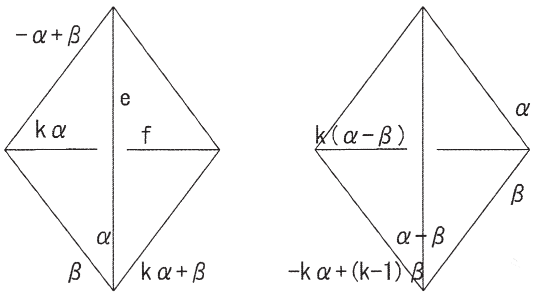

Figure 5: The left is $\mathcal{G}_0(k, 1)$ and the right is $\mathcal{G}_0(-k, k)$ for $k \leq -2$.

isomorphism $\rho : H^2(BT^2) \to H^2(BT^2)$ defined by $\alpha \mapsto \alpha - \beta$ and $\beta \mapsto -k\alpha + (k-1)\beta$ induces the equivalence $\mathcal{G}_0(k, 1) \simeq \mathcal{G}_0(-k, k)$.

We next consider $\mathcal{G}_0(k, -1)$ and $\mathcal{G}_0(k, 2-k)$ as Figure 6. Then, the identity map on two graphs with

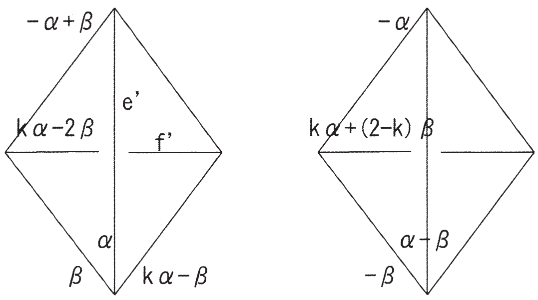

Figure 6: The left is $\mathcal{G}_0(k, -1)$ and the right is $\mathcal{G}_0(k, 2-k)$ for $k \geq 4$.

the isomorphism defined by $\alpha \mapsto \alpha - \beta$ and $\beta \mapsto -\beta$ induces the equivalence $\mathcal{G}_0(k, -1) \simeq \mathcal{G}_0(k, 2-k)$.

We next claim that $\mathcal{G}_0(k, 1) \not\simeq \mathcal{G}_0(k', -1)$ for any integers k, k' such that $|k|, |k'| \geq 2$ (also see the left graphs in Figure 5 and Figure 6). Note that if $|k| \geq 2$ (resp. $|k'| \geq 2$), there are exactly two $(2, 2)$-type GKM subgraphs in $\mathcal{G}_0(k, 1)$ (resp. $\mathcal{G}_0(k', -1)$), say K_1 and K_2 (resp. K_1' and K_2'). Moreover, both of $e = K_1 \cap K_2$ and $e' = K_1' \cap K_2'$ are the edge whose axial function is labelled by α (see Figure 5 and

Figure 6). Assume that there is an equivalent map $\mathcal{G}_0(k,1) \simeq \mathcal{G}_0(k',-1)$. Then, this equivalent map preserves $(2,2)$-type GKM subgraphs; therefore, $e \mapsto e'$ and also $f \mapsto f'$ (they are edges in the twisted position with e and e', see Figure 5 and Figure 6). Hence, this shows that every equivalent map satisfies $\alpha \mapsto \pm\alpha$ and $k\alpha \mapsto \pm(k'\alpha - 2\beta)$. However, this gives a contradiction because if $\alpha \mapsto \pm\alpha$ then $k\alpha \mapsto \pm k\alpha$. Consequently we have that $\mathcal{G}_0(k,1) \not\simeq \mathcal{G}_0(k',-1)$.

We finally claim that $\mathcal{G}_0(1,1) \simeq \mathcal{G}_0(1,-1) \simeq \mathcal{G}_0(-1,1) \not\simeq \mathcal{G}_0(-1,-1)$. Because $\mathcal{G}_0(m,n) \simeq \mathcal{G}_0(n,m)$, we have that

$$\mathcal{G}_0(1,-1) \simeq \mathcal{G}_0(-1,1).$$

See Figure 7. In Figure 7, by taking the isomorphism $\alpha \mapsto -\alpha + \beta$ and $\beta \mapsto \beta$, we have that

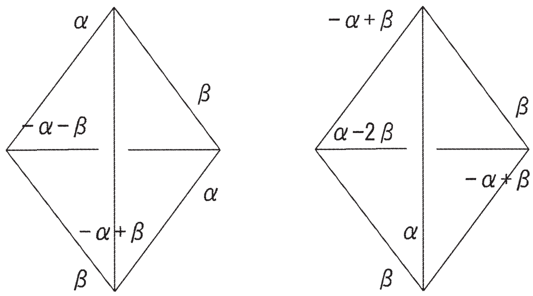

Figure 7: The left is $\mathcal{G}_0(1,1)$ and the right is $\mathcal{G}_0(1,-1)$.

$$\mathcal{G}_0(1,1) \simeq \mathcal{G}_0(1,-1).$$

Note that $\mathcal{G}_0(1,1)$ has two pair of two edges with the same axial functions, i.e., α and β (see the left of Figure 7). On the other hand, $\mathcal{G}_0(-1,-1)$ does not have such pair. If they are equivalent then such pair must be preserved. Hence, we have

$$\mathcal{G}_0(1,1) \not\simeq \mathcal{G}_0(-1,-1).$$

This establishes the statement of this proposition. □

With the method similar to that demonstrated in the proof of Proposition 3.4, we have the following series of propositions:

Proposition 3.5. The $(3,2)$-type GKM graph $(K_4, \alpha, \nabla^{(1)})$ is equivalent to $\mathcal{G}_1(m)$ (Figure 8) for some $m \in \mathbb{N}$; namely, $\mathcal{G}_1(m) \simeq \mathcal{G}_1(m')$ if and only if $|m| = |m'|(\neq 0)$.

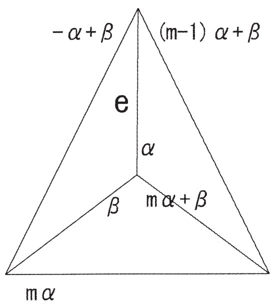

Figure 8: A (3,2)-type GKM graph $\mathcal{G}_1(m)$ $(m \in \mathbb{N})$ with the connection $\nabla^{(1)}$, i.e., $\nabla_e^{(1)} \neq \nabla_e^{st}$. This axial function is nothing but the axial function of $\mathcal{G}_0(m,1)$.

Proposition 3.6. The $(3,2)$-type GKM graph $(K_4, \alpha, \nabla^{(2)})$ is equivalent to $\mathcal{G}_2$ (Figure 9).

Proposition 3.7. The $(3,2)$-type GKM graph $(K_4, \alpha, \nabla^{(3)})$ is equivalent to $\mathcal{G}_3$ (Figure 10).

We also have the following proposition.

Proposition 3.8. For $k = 4,5,6$, there are no axial functions α such that $(K_4, \alpha, \nabla^{(k)})$ is a GKM graph.

Consequently, we establish Theorem 1.1.

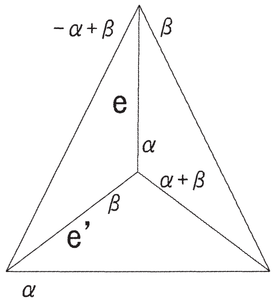

Figure 9: A (3,2)-type GKM graph $\mathcal{G}_2$ with the connection $\nabla^{(2)}$, i.e., $\nabla^{(2)}_e \neq \nabla^{st}_e$ and $\nabla^{(2)}_{e'} \neq \nabla^{st}_{e'}$. This axial function is nothing but the axial function of $\mathcal{G}_0(1,1)$.

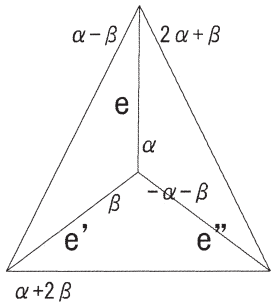

Figure 10: A (3,2)-type GKM graph $\mathcal{G}_3$ with the connection $\nabla^{(3)}$, i.e., $\nabla^{(3)}_e \neq \nabla^{st}_e$, $\nabla^{(3)}_{e'} \neq \nabla^{st}_{e'}$ and $\nabla^{(3)}_{e''} \neq \nabla^{st}_{e''}$. This axial function is not induced from $\mathcal{G}_0(m,n)$ for any $m,n \in \mathbb{Z} \setminus \{0\}$.

4 Final remarks

We finally remark some facts and ask some questions for future works.

4.1 The group of axial functions

We have introduced the *group of axial functions* $\mathcal{A}(\Gamma,\alpha,\nabla)$ for GKM graph (Γ,α,∇) in [Ku2]. The rank of this free abelian group evaluates the maximal axial function which is the extension of α. It is not so difficult to compute them for our GKM graphs in Theorem 1.1. The following is the results of computations.

Theorem 4.1. *The group of axial functions for each GKM graph in Theorem 1.1 is as follows:*

$$\mathcal{A}(\mathcal{G}_0(1,1)) \simeq \mathcal{A}(\mathcal{G}_0(-1,-1)) \simeq \mathcal{A}(\mathcal{G}_0(k,\pm 1)) \simeq \mathbb{Z}^3;$$
$$\mathcal{A}(\mathcal{G}_1(m)) \simeq \mathcal{A}(\mathcal{G}_2) \simeq \mathcal{A}(\mathcal{G}_3) \simeq \mathbb{Z}^2.$$

Therefore, we see that $\mathcal{G}_0(1,1)$, $\mathcal{G}_0(-1,-1)$ and $\mathcal{G}_0(k,\pm 1)$ can be induced from $\mathcal{G}_{st}$.

4.2 GKM cohomology

The most important invariant of GKM graphs is the equivariant cohomology of GKM graph (introduced in [GZ]). We call them a *GKM cohomology* in this paper and denote it as $H^*_T(\Gamma,\alpha,\nabla)$. It is well-known that

$$H^*_T(\mathcal{G}_{st}) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_3,\tau_4]/\langle \tau_1\tau_2\tau_3\tau_4 \rangle,$$

where $\deg \tau_i = 2$ $(i = 1,2,3,4)$, and its $H^*(BT^3)$-algebraic structure is obtained by

$$\alpha \mapsto \tau_1 - \tau_4; \ \beta \mapsto \tau_2 - \tau_4; \ \gamma \mapsto \tau_3 - \tau_4.$$

Because $\mathcal{G}_0(m,n)$ is obtained by changing γ to $m\alpha+n\beta$ (also see Section 4.1), its GKM cohomology is as follows:

$$H_T^*(\mathcal{G}_0(m,n)) \simeq H_T^*(\mathcal{G}_{st})/\mathcal{I},$$

where

$$\mathcal{I} = \langle (\tau_3-\tau_4) - m(\tau_1-\tau_4) - n(\tau_2-\tau_4)\rangle.$$

Note that we do not need to use the connection to define the GKM cohomology, though the connection is essential to define the group of axial functions (see [Ku2]). Therefore,

$$H_T^*(\mathcal{G}_1(m)) \simeq H_T^*(\mathcal{G}_0(m,1)),$$
$$H_T^*(\mathcal{G}_2) \simeq H_T^*(\mathcal{G}_0(1,1)).$$

This establishes that the following theorem:

Theorem 4.2. *The following isomorphisms hold:*

$$H_T^*(\mathcal{G}_0(1,1)) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(\tau_1+\tau_2-\tau_4)\rangle;$$
$$H_T^*(\mathcal{G}_0(-1,-1)) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(-\tau_1-\tau_2+3\tau_4)\rangle;$$
$$H_T^*(\mathcal{G}_0(k,1)) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(k\tau_1+\tau_2-k\tau_4)\rangle;$$
$$H_T^*(\mathcal{G}_0(k,-1)) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(k\tau_1+\tau_2+(2-k)\tau_4)\rangle;$$
$$H_T^*(\mathcal{G}_1(m)) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(m\tau_1+\tau_2-m\tau_4)\rangle;$$
$$H_T^*(\mathcal{G}_2) \simeq \mathbb{Z}[\tau_1,\tau_2,\tau_4]/\langle \tau_1\tau_2\tau_4(\tau_1+\tau_2-\tau_4)\rangle.$$

If we divide them by $\tau_1-\tau_4$ and $\tau_2-\tau_4$, i.e., $H^{>0}(BT^2)$, we obtain the ordinary cohomology counterparts, say $H^*(\Gamma,\alpha,\nabla)$.

Corollary 4.3. The following isomorphisms hold:

$$H^*(\mathcal{G}_0(m,n)) \simeq H^*(\mathcal{G}_1(m)) \simeq H^*(\mathcal{G}_2) \simeq \mathbb{Z}[\tau_4]/\langle \tau_4^4\rangle \simeq H^*(\mathbb{C}P^3).$$

So the following case is still remaining:

Problem 4.4. *Compute $H_T^*(\mathcal{G}_3)$ without using geometry (see [FIM]).*

Remark 4.5. Note that the complex quadric $Q_3 = SO(5)/SO(3)\times SO(2)$ with T^2-action is a GKM manifold whose combinatorial type of GKM graph is K_4. Therefore, by Corollary 4.3 (and $H^*(\mathbb{C}P^3) \not\simeq H^*(Q_3)$ over integer coefficients), we have that the GKM graph $\mathcal{G}_3$ must be obtained from Q_3 with T^2-action. So, by [GKM], $H_T^*(\mathcal{G}_3) \simeq H_T^*(Q_3)$.

In addition, because every $\mathcal{G}_0(m,n)$ is defined by the restricted T^2-action of the standard T^3-action on $\mathbb{C}P^3$ (see Section 4.1). So it is natural to ask whether there are some nice geometric objects for the other GKM graphs $\mathcal{G}_1(m)$, $\mathcal{G}_2$.

Problem 4.6. *Are there any equivariantly formal, simply connected GKM manifolds which define GKM graphs $\mathcal{G}_1(m)$ and $\mathcal{G}_2$?*

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