THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 51 A pp. 1-5(2015)

Gauss codes and Seifert systems of oriented link diagrams

Kanako Suto and Kiyokazu Suto*

Graduate School of Science
**Department of Applied Mathmatics, Faculty of Science,*
Okayama University of Science,
1-1 Ridai-cho, Kita-ku, Okayama 700-0005, Japan

(Received July 16, 2015; accepted November 9, 2015)

For a link diagram, a Gauss code is one of representations of the diagram using crossing labels, and the Seifert system is a plane figure which gives a geometrical characteristic of the link. The number of circles in the system gives an effective way to determine the braid index of the link. In this article, we introduce another metric to investigate the braid index of a link, which is computed in a purely combinatorial manner from the Gauss code of a link diagram, and show that our metric is equally effective as the number of circles in the Seifert system of the diagram.

Keywords: knot, Gauss code, Seifert system, braid index.

1. Introduction

In [4], we showed that the braid index of a link can be determined by counting numbers of strictly increasing maximal subsequences of sequences obtained from Gauss codes of its diagrams, and conjectured that, for each link diagram D, the minimal number $i(D)$ of such subsequences coincides with the number $s(D)$ of Seifert circles.

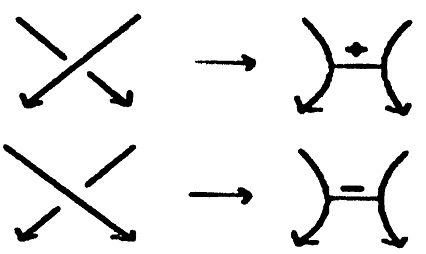
This article is devoted to give an affirmative answer to this conjecture. That is, we show that $i(D)$ is always equal to $s(D)$.

Further, we provide an algorithm to obtain a labeling of crossings of D which gives $i(D)$. The computational quantity of our algorithm is $O(n)$ in both time and space, with n the number of crossings of D.

Since we need to visit all crossings of D to count the number of Seifert circles of D, we can conclude that computing $i(D)$ is equally effective as computing $s(D)$ in the view point of computational quantity.

2. Seifert systems and braid indices

Let L be an oriented link, and D be a diagram of L. By applying the following deformation

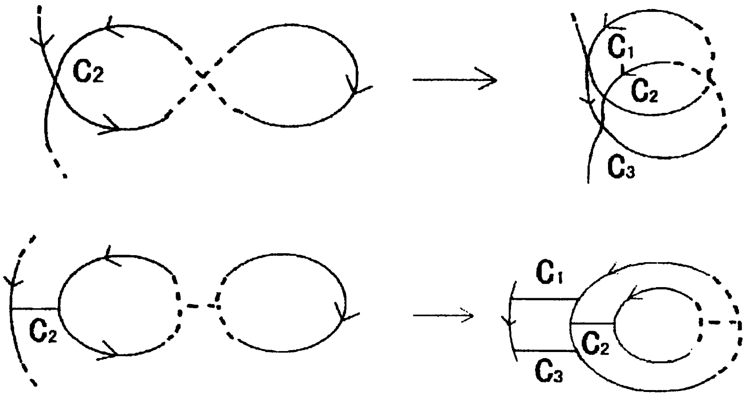


(2.1)

to all the crossings of D, we obtain a set of mutually disjoint oriented circles and lines with signs connecting those circles. This system is called Seifert system of D, each circle in the system a Seifert circle, and each line connecting two Seifert circles a connection. The connections of S are in one-to-one correspondence with the crossings of D. So we identify a connection of S with the corresponding crossing of D throughout this paper. We denote by $s(D)$ the number of Seifert circles of the diagram D after [5].

In [5], it is proved that D can be deformed into the closure of a braid with $s(D)$ strands, and that, as an immediate consequence of the result, the braid index of the link L equals the minimum of $s(D)$'s with D varying all the diagrams of L.

Deformation of D into a closed braid is performed as repetition of a move of a Seifert circle. Each move of a Seifert circle may or may not generate new crossings. If new crossings are generated, they are necessarily of the form



(2.2)

That is, two new crossings c_1 and c_3 per one original crossing c_2 are generated in such a way that the original crossings is placed between new ones along the moved circle.

3. Gauss codes and braid indices

We keep the same notations L and D as in the previous section.

Let $L_1,\cdots, L_m$ be the connected components of L, $D_1,\cdots, D_m$ the respective images in D, $c(D)$ the set of all crossings of D, $n(D)$ the number of elements of $c(D)$, and $f\colon\ c(D) \ni c \mapsto f(c) \in \{1,\cdots, n(D)\}$ a one-to-one mapping.

We choose a point P_i on a strand of D_i different from any crossing for each $1 \leq i \leq n$. We start from P_1 and walk through D_1 after the given orientation until we arrive at P_1 again, and repeat such walks on all $D_2,\cdots, D_m$.

For each crossing c, we pick up the positive integer $f(c)$ when we pass the crossing c along the over strand, or the negative integer $-f(c)$ when we pass c along the under strand while the walk. For each D_i, by arranging the integers for all crossings after the walk, we get a sequence of integers. The list of these sequences for $D_1,\cdots, D_m$ is called a Gauss code of D.

We consider the list of the sequences of positive integers obtained by replacing all integers in the Gauss code with their absolute values. Each sequence splits into maximal strictly increasing subsequences. Let $i(D; f;P_1,\cdots, P_m)$ be the sum of the numbers of theses subsequences, $i(D)$ be the minimum of $i(D;f;P_1,\cdots, P_m)$ for all mappings f and all choices of starting points $P_1,\cdots, P_m$, and $i(L)$ be the minimum of $i(D)$'s for all diagrams D's of L.

A Seifert circle C of D is divided into one or more oriented arcs by connections. Let a be one of such arcs, c the crossing corresponding to the start point of a, and c' the crossing corresponding to the end point of a. We call a increasing if $f(c) < f(c')$.

Let $b(L)$ be the braid index of L. In [4], by counting non-increasing arcs in each Seifert circle, we showed the following theorem.

Theorem 3.1. $i(L)$ *equals* $b(L)$ *for any oriented link* L.

4. The minimal number of maximal increasing sequences of a Gauss code

In [4], adding to the result described in the previous section, we reported about our computations of $i(D)$'s for diagrams D's in the well-known table of [1]: they are all the same as $s(D)$'s. And further we conjectured that this result will be extended to an arbitrary oriented link diagram.

We now show that the answer to this conjecture is affirmative.

For any oriented link diagram D, as described in §2, by applying moves of the form (2.2) to the Seifert system S of D, we obtain a Seifert system S' in which all Seifert circles share a single point O as their center and have the same orientation around O. We arrange the connections in S' in such a way that any two connections are placed on two different half lines starting from O.

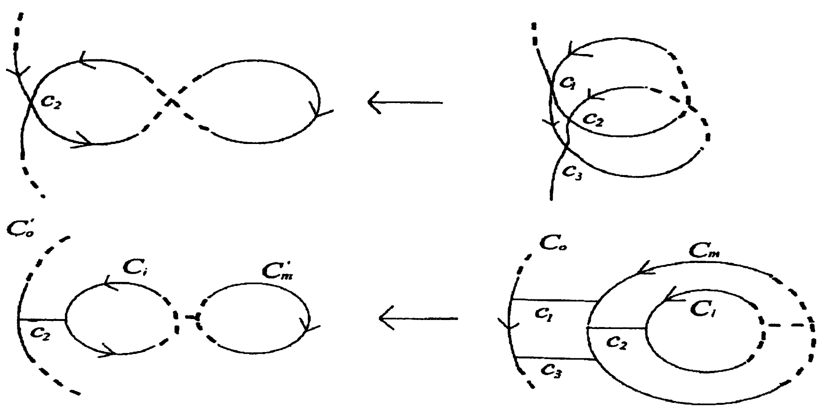
We choose an arbitrary crossing c_0 of D, and consider a half line including the connection c_0 in S' and starting from O.

We rotate this half line around O until it meets the starting connection again, and assign the integers 1,2,... in order to all the crossings after this rotation. We denote the assigned integer by $f(c)$ for each crossing c of D'. Obviously each circle in S' has one and only one non-increasing arc.

We consider the inverse deformation from D' to D, and show below that the number of non-increasing arcs does not change in each step of this deformation.

We denote by C_i, C_m, and C_0, the inner circle, the moved circle, and the outer circle in each inverse move of (2.2), repectively.

For notational convenience, we denote by C'_m and C'_0, the circles C_m and C_0 after the inverse move respectively:



(4.1)

In the move, crossings c_1 and c_3 disappear, and c_2 is newly connected to C'_0. Hence the connections along C_i do not change, and those along C'_m simply reduce. And so it is clear that each of these circles still has one and only one non-increasing arc.

We investigate the circle C'_0.

In case $f(c_1) < f(c_3)$, the non-increasing arc of C_m must be included in the arc from c_3 to c_1, Since the number of non-increasing arcs of C_m is 1, it must hold that

$$f(c_1) < f(c_2) < f(c_3) \tag{4.2}$$

In case $f(c_1) > f(c_3)$, since the number of non-increasing arcs of c_0 is 1, it holds that

$$f(c_3) < f(c) < f(c_1) \quad \text{for any connection } c \text{ along } C_0 \tag{4.3}$$

Since again the number of non-increasing arcs of C_m is 1, there are just two possibilities:

$$f(c_1) > f(c_2) \tag{4.4}$$

or

$$f(c_2) > f(c_3) \tag{4.5}$$

If (4.4) holds, it holds that $f(c_3) \leq f(c) \leq f(c_2)$ for any connection c along C_m.
Hence thanks to (4.3)

$$f(c_2) \geq f(c) \text{ for any connection } c \text{ along } C_m \text{ or } C_0 \tag{4.6}$$

If (4.5) holds, it holds that $f(c_2) \leq f(c) \leq f(c_1)$ for any connection c along C_m.
Hence thanks again to (4.3)

$$f(c_2) \leq f(c) \text{ for any connection } c \text{ along } C_m \text{ or } C_0 \tag{4.7}$$

Let a be the non-increasing arc of C_0, c the crossing at the start of a, and c' the crossing at the end of a.

If both c and c' remain with C'_0, any pair of disappearing crossings c_1 and c_3 in the above argument must satisfy that $f(c_1) < f(c_3)$. And so (4.2) holds for any crossing c_2 newly connected to C'_0.

Let $(c_4,\ c_5)$ be the pair of two crossings other than (c, c') along C'_0 contiguous in this order.

If both c_4 and c_5 were connected to C_0, the arc of C_0 between these crossings must be increasing. Hence it holds that $f(c_4) < f(c_5)$.

If both c_4 and c_5 are crossings newly connected to C'_0, there must be crossings c_6 and c_7 contiguous in this order between c_4 and c_5 along C_0, but disappearing after the move. In this case, it holds that $f(c_4) < f(c_6) < f(c_7) < f(c_5)$.

If c_4 is newly connected to C'_0 but c_5 is not, there must be a crossing c_6 connecting to C_0 and succeeding c_4, but disappearing after the move. In this case, it holds that $f(c_4) < f(c_6) < f(c_5)$.

If c_5 is a crossing newly connected to C'_0 but c_4 is not, there must be a crossing c_7 connecting to C_0 and preceding c_5, but disappearing after the move. In this case, it holds that $f(c_4) < f(c_7) < f(c_5)$.

In all cases above, the arc of C'_0 between c_4 and c_5 is increasing.

Thus a remains to be the only non-increasing arc of C'_0, and hence the number of non-increasing arcs of C'_0 is still 1.

Next we consider the case in which both c and c' disappear after the move.

By a similar argument as above, we see that the sequence A of the crossings along the arc from c' to c of C'_0, is ascending with respect to the mapping f.

If there is no crossing newly connected to the arc from c to c' of C'_0, the arc from the last member of A to the first member A is the only non-increasing arc of C'_0.

If there is a crossing c'' newly connected to the arc from c to c' of C'_0, thanks to (4.6) and (4.7), it holds that

$$f(c'') > f(c)\ \forall c \in A \tag{4.8}$$

or

$$f(c'') < f(c)\ \forall c \in A \tag{4.9}$$

If (4.8) holds, the arc from c'' to the first member of A is the only non-increasing arc of C'_0.

If (4.9) holds, the arc from the last member of A to c'' is the only non-increasing arc of C'_0.

By similar arguments, we see that the number of non-increasing arcs of C'_0 is 1 also in the case where just one of c and c' disappears after the move.

Thus we have the following lemma.

Lemma 4.1. *For any oriented link diagram D and any crossing c_0 of D, there exists a numbering of crossings of D such that each Seifert circle has one and only one non-increasing arc, and that the number assigned to c_0 is the minimum.*

If starting points $P_1,\cdots,P_m$ are taken in such a way that the first crossing of D_i is minimal among the possible choices with respect to f for each $1 \leq i \leq m$, $i(D; f; P_1,\cdots,P_m)$ is equal to the number of all non-increasing arcs. Hence we have the following theorem as an immediate consequence of the above lemma.

Theorem 4.2. *$i(D)$ equals $s(D)$ for any oriented link diagram D.*

5. An algorithm to minimize the number of maximal increasing subsequences of a link diagram

Let D be a link diagram of an oriented link as in the previous sections.

Choose a crossing c_0 of D, and a Seifert circle C_0 of D to which c_0 connects.
We define an order of all crossings of D inductively, with respect to which c_0 is the minimum.

First, let $c_1,\cdots,c_p$ be the rest of the crossings connecting to C_0, where indices $0,1,\cdots,p$ are assigned after the orientation of C_0. We define the order of these crossings as $c_i < c_j$ if and only if $i < j$.

Assume that we have defined orders of all crossings connecting to Seifert circles $C_0,\cdots,C_q$, and that there is a crossing connecting to one of these circles and to a Seifert circle different from any of $C_0,\cdots,C_q$.

Let c' be the minimal one among such crossings, and C_{q+1} the Seifert circle other than $C_0,\cdots,C_q$, to which c' connects. We define an order of the crossings connecting to C_{q+1} with respect to which c' is the minimum among the crossings, by the same manner as for C_0.

There may exist crossings $c'_1 < c'_2$ connecting to C_{q+1} such that each of them connects to one of $C_0,\cdots,C_q$. However, thanks to Lemma 4.1, the new order of these crossings does not conflict with the old one.

After completing the above process, we can define a mapping f of $c(D)$ to $\{1,\cdots,n(D)\}$ in such a way that $f(c) < f(c') \Leftrightarrow c < c'$.

This process can be easily coded with any programing language which supports one dimensional arrays. A sample program is available at [3]. In this program, three one-dimensional arrays are used. The first one represents the link diagram D, the second one is used as a first-in-last-out buffer to keep the crossings to be processed, and the last one expresses the avobe mentioned order of the crossings.

Since the number of members of these arrays are proportional to $n = n(D)$ as seen from the program, the computational quantity in space is $O(n)$. Further each crossing is processed within a constant time, so the computational quantity in time is also $O(n)$.

Refernces

[1] Rolfsen, D., Knots and Links, Mathmatics Lecture Series 7, Publish or Perish, Cambridge, 1976.

[2] Sutou, K., Generation and calculation of Jones polynomials of knots with eleven crossings, The Bulletin of the Okayama University of Science, No.43 A, 2007, pp.1-6.

[3] Suto, K., http://www.merry.xmath.ous.ac.jp/math/minimum_labeling/.

[4] Suto, K., Suto, K., and Mizobuchi, N., A new metric to measure the braid index of a link, The Bulletin of the Okayama University of Science, No. 48 A, 2012, pp.1-4.

[5] Yamada, S., The minimal number of Seifert circles equals the braid index of a link, Invent. math, 89, 1987, pp. 347-356.