THE BULLETIN OF OKAYAMA UNIVERSITY OF SCIENCE No. 50 A pp. 9-12(2014)

A planarity condition of chord diagrams for graphs with bridges

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(Received September 29, 2014; accepted November 6, 2014)

Fleming and Mellor extended the notion of chord diagrams on circles to general graphs, and gave planarity conditions of them in various cases. We found a flaw in the proof of the case where the base graphs are directed and have bridges. We shall correct the proof, and give a similar condition in the non-directed case.

Keywords: graph; chord diagram; planarity.

Introduction

In [1], Fleming and Mellor extended the notion of chord diagrams on circles to general graphs, aiming to provide a new tool for spatial graph theory. One of the main goals of the article, is to discuss about planarity of chord diagrams for planar graphs. They gave some planarity conditions in various cases. We found a flaw in the proof of the case where the base graphs are directed and have bridges. In this article, we shall correct the proof, and show a condition for planarity of non-oriented chord diagrams for non directed graphs. Further we will discuss about the computational quantity to examine our condition.

1. Chord diagrams for graphs

Let $G = (E,V)$ be a graph with edge set E and vertex set V. In the definition below, we regard G as a one dimensional complex with the set E of one dimensional simplexes and with the set V of 0-dimentional simplexes.

A chord diagram $D = (C,\varepsilon)$ on G of degree n is a pair of n-elements set $C = \{c_1 \dots, c_n\}$, and a map ε of C into $2^{G \setminus V}$ where, for each $c \in C$, the image $\varepsilon(c)$ is two elements subset of G \ V. We call an element $c \in C$ a chord of D, and the elements in $\varepsilon(c)$ the endpoints of c. We further require that, for any $c,c' \in C$, if $c \neq c'$ then it holds that $\varepsilon(c) \cap \varepsilon(c') = \emptyset$.

When G is a directed graph, an oriented chord diagram D is a triplet(C,ε,σ), where (C,ε) is the same as above, and σ is a map of $\cup_{c\in C} \varepsilon(c)$ into the set $\{-1,1\}$. For $c \in C$, the two endpoints A and B of c is said to have the same orientation if $\sigma(A)=\sigma(B)$, or the opposite orientation otherwise.

These definitions are slightly different from those in [1], but essentially the same. If we take a loop (i.e. a graph with just one edge and just one vertex) as G, we obtain an equivalent for a classical chord diagram on a circle.

2. Intersection graphs of chord diagrams for graphs

Let $G = (E,V)$ be a graph which may be directed, D be a chord diagram on G. For a chord c of D, we denote by $P(c)$ the set of all non-directed paths in G which connect two endpoints of c.

Let c' be an another chord of D. If any paths $p \in P(c)$ and $p' \in P(c')$ overlap, but none of them properly included in the another, we say that c and c' intersect. If D is a classical chord diagram on a loop G drawn in plane as line segments connecting points of a circle inside it, this definition indicates that c and c' geometrically intersect.

We consider a simple graph $\Gamma_G(D)$, the vertex set of which is the set of chords of D. Vertices c and c' of $\Gamma_G(D)$ are adjacent if and only if they intersect as chords of D. We call this graph the intersection graph of D.

3. Chord diagrams for planar graphs

From now on, we concentrate on chord diagrams for planar graphs.

Let G be a planar graph, D be a chord diagram for G, and f be an embedding of G into the plane $\mathbf{R}^2$.

For a chord c of D with endpoints on edges e_1 and e_2 of G, we say that c respects f if the following conditions are satisfied:

(1) The edges e_1 and e_2 lie on the boundary of a single region R of $f(G)$ (i.e. one of connected components of $\mathbf{R}^2 \setminus f(G)$).

(2) This condition is required only when G is directed, D is oriented, and none of e_1 or e_2 is a bridge. If the above region R is on the same side of e_1 and e_2 with respect to the direction inherited from that of G, then the endpoints of c have opposite orientation. Otherwise the endpoints of c have the same orientation.

When G is directed and the edges e_1 and e_2 lie on the boundary of two different regions, any of the regions is always on the opposite side of each edge to the another region. So whether or not the condition (2) is satisfied, is independent from the choice of a region satisfying the condition (1).

If all chords of D respect f, we say that D respects f.

We now define a graph $\Gamma_G(D;f)$ resembling $\Gamma_G(D)$ but the adjacency condition is somewhat relaxed by using f. The vertex set of $\Gamma_G(D;f)$ is the set of chords of D. Vertices c and c' of $\Gamma_G(D;f)$ are adjacent if and only if, for any path p connecting the endpoints of c on the boundary of a single region of $f(G)$ and any path p' connecting the endpoints of c' on the boundary of a single region of $f(G)$, p and p' overlap, but none of them included in the another. We call $\Gamma_G(D;f)$ the intersection graph of G with respect to f.

If two chords are adjacent in $\Gamma_G(D)$, they are obviously adjacent in $\Gamma_G(D;f)$. Hence we can regard $\Gamma_G(D)$ as a subgraph of $\Gamma_G(D;f)$.

Let φ be an injective map φ from the set of chords of D onto a set of piecewise smooth curve segments in the plane $\mathbb{R}^2$. We call φ embedding of D into the plane extending f, if φ satisfies the following three conditions:

(1) For each chord c of D, $\varphi(c)$ is a curve inside a region of $f(G)$ connecting the images of endpoints of c by f.

(2) For two different chords c and c' of D, $\varphi(c) \cap \varphi(c') = \emptyset$.

(3) When G is directed and D is oriented, for any chord c of D, edges e_1 and e_2 in $f(G)$ on which endpoints of $\varphi(c)$ lie, and the region R inside which $\varphi(c)$ is, the endpoints of c have the opposite orientation if R is on the same side of e_1 and e_2 with respect to the directions inherited from G, or they have the same orientation otherwise.

4. Planarity of chord diagrams for planar graphs without bridges

In the case of the graph without bridge, we have the following planarity condition[1, Proposition 3]

Theorem 4.1. *Let G be a planar graph without bridges, f be an embedding of G into $\mathbb{R}^2$, and D be a chord* *diagram for G. Then f extends to an embedding of D if and only if the following conditions are satisfied:*

(1) D respects f.

(2) The chords of D are labeled by regions of f(G) in such a way that:

- *If a chord c of D is labeled by a region R of f(G), the images of endpoints of c by f are on the boundary of R.*
- *For a region R of f(G), if two chords c and c' of D are both labeled by R, then c and c' are not adjacent in $\Gamma_G(D;f)$.*

Note that, although the above condition is restricted to the case where G is directed and D is oriented in [1], their proof is obviously applicative to the non-directed case.

5. Planarity of chord diagrams for planar graphs with bridges

As before, let G be a planar graph, f be an embedding of G into $\mathbf{R}^2$, and D be a chord diagram for G.

For an edge e of G which is not a loop, by replacing e with two new edges e' and e'' connecting the same endpoints as e, we obtain a new graph. We denote by G_e the new graph, and call it the blowup of e in G. If G is directed, we give the same direction to the new edges as e.

For a subset S of the edge set of G, we denote by G_S the graph obtained by blowing up all the edges in S. It is obvious that f is naturally extended to an embedding of G_S into $\mathbf{R}^2$. Let c be a chord of D, and k be the number of endpoints of c which lie on one of edges in S. If $k \geq 1$, we copy such endpoints to the newly added edges in G_S. We wish to add new chords to D connecting these new endpoints in the same manner as c. Though there are 2^k combinations of new endpoints which the new chords can connect, because of the requirement that all endpoints of chords must be different from each other, the combinations are divided to two groups, each of which contain 2^{k-1} pair of new possible endpoints. We choose one of the groups and replace c with new chords connecting the pairs of endpoints in the chosed group. When D is oriented, we give the same orientation to new endpoints as the original endpoints. We denote by $\beta(c)$ this chosed group when $k \geq 1$, and the pair of endpoints of c when $k = 0$. Repeating these choices for all chords, we obtain a new chord diagram. We denoted it by D_S^β.

We consider a new graph $\overline{\Gamma_{G_S}(D_S^\beta;f)}$ containing the intersection graph $\Gamma_{G_S}(D_S^\beta;f)$ as a spanning subgraph. Two chords c and c' of D_S^β are adjacent in $\overline{\Gamma_{G_S}(D_S^\beta;f)}$ if and only if they are adjacent in $\Gamma_{G_S}(D_S^\beta;f)$ or came from the same chord of D.

Taking the set of all the bridges of G as S, we obtain the following criterion for planarity of D.

Theorem 5.1. *Let G be a planar graph, f be an embedding of G into $\mathbb{R}^2$, B be the set of all the bridges of G, D be a chord diagram for G, R_0 be the union of the regions inside two edges came from blowups of the elements in B, R_1 be the unbound region of $f(G_B)$, and $R_2, \dots, R_n$ $(n \geq 2)$ be all the bounded regions of $f(G_B)$ which are not included in R_0. Then f extends to an embedding of D if and only if the following conditions are satisfied:*

(1) *D respects f.*

(2) *There exists an above mentioned choice β for B such that the chords of D_B^β are labeled by $R_0, R_1, \dots, R_n$ in such a way that:*

- *If a chord c of D_B^β is labeled by R_i, the images of endpoints of c are on the boundary of R_i.*

- *If two chords c and c' of D_B^β are both labeled by R_i, then c and c' are not adjacent in $\overline{\Gamma_{G_B}(D_B^\beta; f)}$.*

- *If two chords of D_B^β are came from one chord of D by blowing up of a bridge, one of them is labeled by R_0, and the another is labeled by R_1.*

Proof. We first show the sufficiency. By removing all the chords labeled by R_0 from D_B^β, we obtain a chord diagram for G_B satisfying all the conditions in Theorem 4.1. Hence f extends to a planar embedding of this chord diagram. Chords in this diagram are in natural one-to-one correspondence with those in D, and further, they are all drawn outside R_0 in the embedding. So, collapsing R_0 to line segments, we naturally get a planar embedding of D.

The proof of the necessity is essentially the same as that of [1,Proposition 5], noting that an embedding of D automatically determines one choice β. Q.E.D.

Note that the conditions in our theorem requires that D_B^β respect f. When G is directed and D is oriented, this requirement uniquely determines the choice β. With this choice, D_B^β gives the reduced oriented chord diagram for G_B, and hence our theorem gives [1, Proposition 5] as a special case.

In the proof of sufficiency of [1, Proposition 5], it is mentioned that f extends to a planar embedding of the reduced oriented diagram. But there exist counter examples to this statement. One simplest example is single chord connecting two consecutive line segments.

As mentioned in [1], if G is directed and D is oriented, this criterion is examined in polynomial time.

In the non-oriented case, we do not have any idea to determine effectively which choice β gives a planar embedding. We must examine all the 2^N choices with N the number of chords, one of whose endpoints lies on a bridge.

References

1) T. Fleming and B. Mellor, Chord diagrams and Gauss codes for graphs, ArXiv preprint,arXiv:math/0508269v2, 2006.