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Elimination of certain crossings of braids

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We shall show some equations of the form FDG=D in n-braid group B_n with standard generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$. These equations represent certain geometric deformations, and are used in [S] to reduce the number of braids of which Jones polynomials are calculated.

Keywords: braid; braid group; knot.

1. Introduction

In this article we treat the n-braid group B_n with standard generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$. It is well known that B_n is generated under fundamental relations representing some geometric deformations.

We extend the relations to equations which represent more complicated deformations. These equations are useful in computer programs [S] to generate knots inductively and to calculate Jones polynomials of generated knots.

2. Braid group

In this section we give definition of braid groups. We fix one rectangular box. On each of the ceiling and the floor of the box, we arrange n points, and label them with the numbers $1, \dots, n$ in order.

An n-braid is a set of n strings of which one joins one point of the floor and that of the ceiling.

Let σ_i be a braid satisfying the following conditions (see Figure 1).

a) A string joins i-th point of the ceiling to (i+1)-th point of the floor, an other string joins (i+1)-th point of the ceiling to i-th point of the floor, and the former crossing over the latter.

b) σ_i has just one crossing described in a).

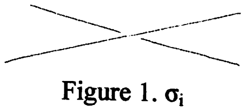

Figure 1. σ_i

Two braids are called of the same type, when we can deform one of the braids to the another, without tearing strings.

For braids b_1 and b_2, we obtain new braid b by connecting the point of the floor of b_1 and the point of the ceiling of b_2 in numeric order. We call this b the product of b_1 and b_2 (see Figure 2).

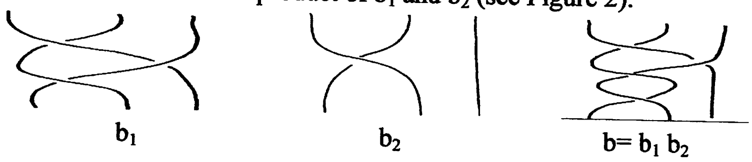

Figure 2.

Let B_n be all the n-braids where we identify n-braids of the some type. B_n is a group with the above products.

This group is called a braid group. The group B_n is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$.

In this article we use the following notations. For braids $X_1, X_2, \ldots, X_{v-1}, X_v$, $\prod_{i=1}^{v} X_i$ denotes the product $X_1 X_2 \ldots X_{v-1} X_v$. We fix the notations

(2.1) $$\varepsilon = \pm 1, m = \pm 1, i = 1, \ldots, n-1, j = 1, \ldots, n-1.$$

3. The fundamental relations of the braid group

The relations (3.1) and (3.2) below represent the geometric deformations of the braid as Figure 3.

(3.1) $$\sigma_i^m \sigma_{i+1}^m \sigma_i^m = \sigma_{i+1}^m \sigma_i^m \sigma_{i+1}^m,$$

(3.2) $$\sigma_i^\alpha \sigma_{i+1}^m \sigma_i^{-\alpha} = \sigma_{i+1}^{-\alpha} \sigma_i^m \sigma_{i+1}^\alpha, \ \alpha = \pm 1.$$

We have further

(3.3) $$\sigma_i^{\varepsilon_1}\sigma_j^{\varepsilon_2} = \sigma_j^{\varepsilon_1}\sigma_i^{\varepsilon_2}, \ |i-j| > 1, \ (\varepsilon_1, \varepsilon_2) = (\pm 1, \pm 1), (\pm 1, \mp 1),$$

(3.4) $$\sigma_i^\varepsilon \sigma_j^m \sigma_i^m \sigma_j^{-\varepsilon} = \sigma_j^m \sigma_i^m, \ |i-j| = 1.$$

(3.1) (m=1) (3.2) (α=1, m=−1)

Figure 3.

The n-braid group B_n generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with fundamental relations (3.1),(3.2),(3.3), and (3.4) (see Figure 4).

(3.3) (3.4) (ε=1,m=−1)

Figure 4.

The relation (3.4) can be written as follows.

(3.5) $$\sigma_i^\varepsilon \Big(\prod_{A=i+1}^{i+1} \sigma_A^m \sigma_{A-1}^m \Big) \sigma_{i+1}^{-\varepsilon} = \sigma_{i+1}^m \sigma_i^m,$$

(3.6) $$\sigma_i^\varepsilon \Big(\prod_{A=i-1}^{i-1} \sigma_A^m \sigma_{A+1}^m \Big) \sigma_{i-1}^{-\varepsilon} = \sigma_i^m \sigma_{i-1}^m.$$

We extend these equations to more generic form in the next section.

4. Elimination of crossings

In this section, we shall show some equations of the form $\sigma_c^p D \sigma_s^{-p} = D$ with braids D satisfying certain conditions. The equations represent deformations which reduce crossing numbers of certain braids as Figure 5.

Figure 5.

Theorem 4.1. For any positive integer γ, it holds that

(4.1) $$\sigma_i^{\varepsilon}\Big(\prod_{A=i+1}^{i+\gamma}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+\gamma}^{-\varepsilon} = \prod_{A=i+1}^{i+\gamma}\sigma_A^m\sigma_{A-1}^m .$$

Proof. The equation (4.1) is true for γ=1 by (3.5).
Assume (4.1) is true for γ=K, i.e.,

$$\sigma_i^{\varepsilon}\Big(\prod_{A=i+1}^{i+K}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+K}^{-\varepsilon} = \prod_{A=i+1}^{i+K}\sigma_A^m\sigma_{A-1}^m .$$

When γ=K+1, the left hand side of (4.1) becomes

$$\begin{aligned}
\sigma_i^{\varepsilon}\Big(\prod_{A=i+1}^{i+K+1}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+K+1}^{-\varepsilon} &= \sigma_i^{\varepsilon}\Big(\prod_{A=i+1}^{i+K}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+K+1}^m\sigma_{i+K}^m\sigma_{i+K+1}^{-\varepsilon} \\
&= \sigma_i^{\varepsilon}\Big(\prod_{A=i+1}^{i+K}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+K}^{-\varepsilon}\sigma_{i+K+1}^m\sigma_{i+K}^m \\
&\quad \text{(by (3.1) if } m=-\varepsilon \text{ and by (3.2) if } m=\varepsilon\text{)} \\
&= \Big(\prod_{A=i+1}^{i+K}\sigma_A^m\sigma_{A-1}^m\Big)\sigma_{i+K+1}^m\sigma_{i+K}^m \\
&= \prod_{A=i+1}^{i+K+1}\sigma_A^m\sigma_{A-1}^m .
\end{aligned}$$

□

Theorem 4.2. For any positive integer γ, it holds that

(4.2) $$\sigma_i^{\varepsilon}\Big(\prod_{A=i-1}^{i-\gamma}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i-\gamma}^{-\varepsilon} = \prod_{A=i-1}^{i-\gamma}\sigma_A^m\sigma_{A+1}^m .$$

Proof. The equation (4.2) is true for γ=1 by (3.6).
Assume (4.2) is true for γ=K, i.e.,

$$\sigma_i^{\varepsilon}\Big(\prod_{A=i-1}^{i-K}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i+K}^{-\varepsilon} = \prod_{A=i-1}^{i-K}\sigma_A^m\sigma_{A+1}^m .$$

When γ=K+1, the left hand side of (4.2) becomes

$$\begin{aligned}
\sigma_i^{\varepsilon}\Big(\prod_{A=i-1}^{i-(K+1)}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i-(K+1)}^{-\varepsilon} &= \sigma_i^{\varepsilon}\Big(\prod_{A=i-1}^{i-K}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i-(K+1)}^m\sigma_{i-(K+1)+1}^m\sigma_{i-(K+1)}^{-\varepsilon} \\
&= \sigma_i^{\varepsilon}\Big(\prod_{A=i-1}^{i-K}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i-K-1}^m\sigma_{i-K}^m\sigma_{i-K-1}^{-\varepsilon} \\
&\quad \text{(by (3.1) if } m=-\varepsilon \text{ and by (3.2) if } m=\varepsilon\text{)} \\
&= \Big(\prod_{A=i-1}^{i-K}\sigma_A^m\sigma_{A+1}^m\Big)\sigma_{i-K-1}^m\sigma_{i-K}^m
\end{aligned}$$

$$= \prod_{A=i-1}^{i-(K+1)} \sigma_A^m \sigma_{A+1}^m .$$

□

We can also cancel crossings like Figure 6.

Figure 6.

That is

Theorem 4.3.

$$\sigma_i^{\varepsilon}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{-\varepsilon} = \sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}, \tag{4.3}$$

$$\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{m}\sigma_{i+1}^{\mp1}\sigma_{i-1}^{\mp1}\sigma_i^{\mp1} = \sigma_{i+1}^{\mp1}\sigma_{i-1}^{\mp1}\sigma_i^{m}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}, \tag{4.4}$$

$$\sigma_i^{\varepsilon}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\mp1}\sigma_i^{m}\sigma_{i+1}^{\mp1}\sigma_{i-1}^{\pm1}\sigma_i^{-\varepsilon} = \sigma_{i+1}^{\pm1}\sigma_{i-1}^{\mp1}\sigma_i^{m}\sigma_{i+1}^{\mp1}\sigma_{i-1}^{\pm1}, \tag{4.5}$$

where in each equation, all the double signs correspond to each other.

Proof. By (3.3), the left hand side of (4.3) is equals to

$$\sigma_i^{\varepsilon}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{-\varepsilon} = \sigma_i^{\varepsilon}\sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{-\varepsilon}$$

(by (3.1))

$$= \sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i-1}^{\varepsilon}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{-\varepsilon}$$

(by (3.1) if $\varepsilon = \pm 1$ and by (3.2) if $\varepsilon = \mp 1$)

$$= \sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i-1}^{\varepsilon}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{-\varepsilon}\sigma_i^{\pm1}\sigma_{i-1}^{\pm1}$$

(by (3.2) if $\varepsilon = \pm 1$ and by (3.1) if $\varepsilon = \mp 1$)

$$= \sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\varepsilon}\sigma_{i-1}^{-\varepsilon}\sigma_i^{\pm1}\sigma_{i-1}^{\pm1}$$

(by (3.3))

$$= \sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i-1}^{\pm1}$$

$$= \sigma_{i-1}^{\pm1}\sigma_{i+1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}$$

(by (3.2))

$$= \sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}\sigma_i^{\pm1}\sigma_{i+1}^{\pm1}\sigma_{i-1}^{\pm1}$$

(by (3.3)).

And this equals the right hand side of (4.3).

(4.4), (4.5) also hold to reform as the proof of (4.3).

□

Theorem 4.3 can be extended as Figure 7.

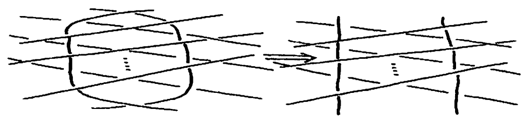

Figure 7.

Thus

Theorem 4.4. For any positive integers γ, $\gamma_{r_1}, \dots, \gamma_{r_5}$, $v_{\gamma s1}, \dots, v_{\gamma s5}$, $p_1, \dots, p_d$, $q_1, \dots, q_f$, satisfying $p_1, \dots, p_d, q_1, \dots, q_f \neq i-1, i, i+1$ and $v_{\gamma s1}, \dots, v_{\gamma s5} \neq i-2, i, i+1, i+2$, the following equations hold

(4.6)
$$\begin{aligned}
&\sigma_i^{\varepsilon}\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{\pm1}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\pm1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big)\sigma_i^{-\varepsilon}\\
&=\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{\pm1}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\pm1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big),
\end{aligned}$$

(4.7)
$$\begin{aligned}
&\sigma_i^{\varepsilon}\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{m}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\mp1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big)\sigma_i^{-\varepsilon}\\
&=\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{m}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\mp1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big)
\end{aligned}$$

(4.8)
$$\begin{aligned}
&\sigma_i^{\varepsilon}\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{m}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\pm1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big)\sigma_i^{-\varepsilon}\\
&=\Big(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}}\Big)\prod_{r=1}^{\gamma}\Big\{\Big(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{\gamma s1}}^{m_{\gamma s1}}\Big)\sigma_{i+1}^{\pm1}\Big(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{\gamma s2}}^{m_{\gamma s2}}\Big)\sigma_{i-1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{\gamma s3}}^{m_{\gamma s3}}\Big)\sigma_i^{m}\\
&\Big(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{\gamma s4}}^{m_{\gamma s4}}\Big)\sigma_{i+1}^{\mp1}\Big(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{\gamma s5}}^{m_{\gamma s5}}\Big)\sigma_{i-1}^{\pm1}\Big\}\Big(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}\Big)
\end{aligned}$$

Proof. By (3.3), the left hand side of (4.6) becomes

$$(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}})\sigma_i^{\varepsilon}\prod_{r=1}^{\gamma}\{(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{rs1}}^{m_{rs1}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})\sigma_{i-1}^{\pm1}(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})\sigma_i^{\pm1}(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})\sigma_{i-1}^{\pm1}\}$$

$$\sigma_i^{-\varepsilon}(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}})$$

$$=(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}})\sigma_i^{\varepsilon}\{(\prod_{s=1}^{\gamma_{11}}\sigma_{v_{rs1}}^{m_{rs1}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})\sigma_{i-1}^{\pm1}(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})\sigma_i^{\pm1}(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})\sigma_{i-1}^{\pm1}$$

$$\cdots$$

$$(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{rs1}}^{m_{rs1}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})\sigma_{i-1}^{\pm1}(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})\sigma_i^{\pm1}(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})\sigma_{i-1}^{\pm1}\}\sigma_i^{-\varepsilon}(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}})$$

$$=(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}})\{(\prod_{s=1}^{\gamma_{11}}\sigma_{v_{rs1}}^{m_{rs1}})(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})$$

$$\cdots$$

$$(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{rs1}}^{m_{rs1}})(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})\}(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}})$$

$$\sigma_i^{\varepsilon}(\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1}\,\sigma_i^{\pm1}\,\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1})\ldots(\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1}\,\sigma_i^{\pm1}\,\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1})\sigma_i^{-\varepsilon}$$

(by (3.3))

$$=(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}})\{(\prod_{s=1}^{\gamma_{11}}\sigma_{v_{rs1}}^{m_{rs1}})(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})$$

$$\cdots$$

$$(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{rs1}}^{m_{rs1}})(\prod_{s=1}^{\gamma_{12}}\sigma_{v_{rs2}}^{m_{rs2}})(\prod_{s=1}^{\gamma_{13}}\sigma_{v_{rs3}}^{m_{rs3}})(\prod_{s=1}^{\gamma_{14}}\sigma_{v_{rs4}}^{m_{rs4}})(\prod_{s=1}^{\gamma_{15}}\sigma_{v_{rs5}}^{m_{rs5}})\}(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}})$$

$$(\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1}\,\sigma_i^{\pm1}\,\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1})\ldots(\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1}\,\sigma_i^{\pm1}\,\sigma_{i+1}^{\pm1}\,\sigma_{i-1}^{\pm1})$$

(by Proof of Theorem 4.3)

$$=(\prod_{\theta_1=1}^{d}\sigma_{p_{\theta_1}}^{m_{\theta_1}})\prod_{r=1}^{\gamma}\{(\prod_{s=1}^{\gamma_{r1}}\sigma_{v_{rs1}}^{m_{rs1}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{r2}}\sigma_{v_{rs2}}^{m_{rs2}})\sigma_{i-1}^{\pm1}(\prod_{s=1}^{\gamma_{r3}}\sigma_{v_{rs3}}^{m_{rs3}})\sigma_i^{\pm1}$$

$$(\prod_{s=1}^{\gamma_{r4}}\sigma_{v_{rs4}}^{m_{rs4}})\sigma_{i+1}^{\pm1}(\prod_{s=1}^{\gamma_{r5}}\sigma_{v_{rs5}}^{m_{rs5}})\sigma_{i-1}^{\pm1}\}(\prod_{\theta_2=1}^{f}\sigma_{q_{\theta_2}}^{m_{\theta_2}}).$$

Thus we obtained the right hand side of (4.6). (4.7) and (4.8) are proved in the similar way. □

We now show equations representing Figure 8.

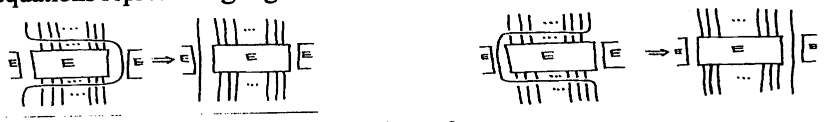

Figure 8.

Theorem 4.5. For any positive integers γ and $t_1, \ldots, t_\lambda$, it holds that

(4.9) $$\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\left(\prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right) = \prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}$$

if $t_j \neq i-1, i, i+\gamma, i+\gamma+1$ for all j.

Proof. When $t_1 \leqq i-2$ or $i+\gamma+2 \leqq t_1$, the left hand side of (4.9) becomes

$$\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\sigma_{t_1}^{\mu_1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right) = \sigma_{t_1}^{\mu_1}\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

(by (3.3)).

When $i+1 \leqq t_1 \leqq i+\gamma-1$ and $\beta = t_1 - i$, the left hand side of (4.9) becomes

$$\sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}\cdots\sigma_{i+\beta-1}^{\pm 1}\sigma_{i+\beta}^{\pm 1}\sigma_{i+\beta+1}^{\pm 1}\sigma_{i+\beta+2}^{\pm 1}\cdots\sigma_{i+\gamma}^{\pm 1}\sigma_{t_1}^{\mu_1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

$$= \sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}\cdots\sigma_{i+\beta-1}^{\pm 1}\sigma_{i+\beta}^{\pm 1}\sigma_{i+\beta+1}^{\pm 1}\sigma_{t_1}^{\mu_1}\sigma_{i+\beta+2}^{\pm 1}\cdots\sigma_{i+\gamma}^{\pm 1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

(by (3.3))

$$= \sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}\cdots\sigma_{i+\beta-1}^{\pm 1}\sigma_{i+\beta+1}^{\mu_1}\sigma_{i+\beta}^{\pm 1}\sigma_{i+\beta+1}^{\pm 1}\sigma_{i+\beta+2}^{\pm 1}\cdots\sigma_{i+\gamma}^{\pm 1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

(by (3.1) if $\mu_1=\pm 1$ and by (3.2) $\mu_1=\mp 1$)

$$= \sigma_{i+\beta+1}^{\pm 1}\sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}\cdots\sigma_{i+\beta-1}^{\pm 1}\sigma_{i+\beta+1}^{\mu_1}\sigma_{i+\beta}^{\pm 1}\sigma_{i+\beta+2}^{\pm 1}\cdots\sigma_{i+\gamma}^{\pm 1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

(by (3.3)

$$= \sigma_{t_1}^{\mu_1}\sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}\cdots\sigma_{i+\beta-1}^{\pm 1}\sigma_{i+\beta+1}^{\mu_1}\sigma_{i+\beta}^{\pm 1}\sigma_{i+\beta+2}^{\pm 1}\cdots\sigma_{i+\gamma}^{\pm 1}\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

$$= \sigma_{t_1}^{\mu_1}\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right).$$

Thus the left hand side of (4.9) is equal to

$$\sigma_{t_1}^{\mu_1}\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\left(\prod_{j=2}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right).$$

Applying the same argument to $t_2, t_3, \ldots, t_\lambda$, we have

$$\left(\prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\pm 1}\right)\left(\prod_{A=i+\gamma}^{i}\sigma_A^{\mp 1}\right)$$

$$= \prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}.$$

□

By the same arguments, we have

Theorem 4.6. For any positive integers γ and $t_1, \ldots, t_\lambda$, it holds that

$$(4.10)\qquad \left(\prod_{A=i+\gamma}^{i}\sigma_A^{\pm 1}\right)\left(\prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}\right)\left(\prod_{A=i}^{i+\gamma}\sigma_A^{\mp 1}\right) = \prod_{j=1}^{\lambda}\sigma_{t_j}^{\mu_j}$$

if $i-1>t_j$, $i+\gamma+1<t_j$ for all j.

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