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A note on a condition for the obstruction ideal of an element α to be equal to the obstruction ideal of a linear fractional transform of α

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Abstract

Let α be an algebraic element over the quotient field of an integral domain R. Let β be a linear fractional transform of α, that is,

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, \ a\alpha - b \neq 0).$$

We give a condition that two obstruction ideals $J_{[\alpha]}$ and $J_{[\beta]}$ are the same under the assumption $(a, b, c, d)R = R$.

Keywords: obstruction ideal of flatness; linear fractional transform; generalized fractional ideal.

Let R be an integral domain with quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with deg $\varphi_\alpha(X) = t$, and write:

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t, \ (\eta_1, \ldots, \eta_t \in K).$$

We define the generalized fractional ideal of α: $I_{[\alpha]} = \bigcap_{i=1}^t I_{\eta_i}$ where $I_{\eta_i} = (R :_R \eta_i) = \{c \in R; \ c\eta_i \in R\}$. We define the obstruction ideal of flatness: $J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \ldots, \eta_t)$ where $(1, \eta_1, \ldots, \eta_t)$ is the R-module generated by $1, \eta_1, \ldots, \eta_t$.

Let β be a linear fractional transform of α, that is,

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, \ a\alpha - b \neq 0).$$

We denote by R^* the set of units of R and set $\Delta_\beta = ad - bc$.

Set $\varphi_\alpha(X, Y) = X^t \varphi_\alpha(Y/X)$. If Δ_β is in R^*, then it is easily verified that

$$\varphi_\beta(X) = \varphi_\alpha(a, b)^{-1} \varphi_\alpha(aX - c, bX - d).$$

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Our notation is standard and our general reference for unexplained terms is [3].

Let $\mathfrak{p}$ be an element of Spec$R$ and α an algebraic element of degree t over the quotient field of R. Set

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t,\ (\eta_1, \ldots, \eta_t \in K)$$

and $(R_\mathfrak{p} :_{R_\mathfrak{p}} \eta_i) = \{c \in R_\mathfrak{p};\ c\eta_i \in R_\mathfrak{p}\}$. Set $I_{R_\mathfrak{p},[\alpha]} = \bigcap_{i=1}^t (R_\mathfrak{p} :_{R_\mathfrak{p}} \eta_i)$ and $J_{R_\mathfrak{p},[\alpha]} = I_{R_\mathfrak{p},[\alpha]}(1, \eta_1, \ldots, \eta_t)_{R_\mathfrak{p}}$ where $(1, \eta_1, \ldots, \eta_t)_{R_\mathfrak{p}}$ is the $R_\mathfrak{p}$-module generated by $1, \eta_1, \ldots, \eta_t$.

Lemma 1. (cf.[1, Lemma 1.1]) *Let R be an integral domain and α an algebraic element over the quotient field of R. Let $\mathfrak{p}$ be an element of* Spec$R$. *Then $I_{R_\mathfrak{p},[\alpha]} = I_{[\alpha]}R_\mathfrak{p}$ and $J_{R_\mathfrak{p},[\alpha]} = J_{[\alpha]}R_\mathfrak{p}$.*

Lemma 2. ([2, Theorem 19]) *Let R be a Noetherian domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

with $\Delta_\beta = ad - bc \neq 0$. Assume that the following conditions hold:

(1) *a is a unit of R.*

(2) $\Delta_\beta + \varphi_\alpha(a,b)I_{[\alpha]} = R$.

Then $J_{[\alpha]} = J_{[\beta]}$.

Lemma 3. ([2, Theorem 22]) *Let R be a Noetherian domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

with $\Delta_\beta = ad - bc \neq 0$. Assume that the following conditions hold:

(1) *b is a unit of R.*

(2) $\Delta_\beta + \varphi_\alpha(-a,-b)I_{[\alpha]} = R$.

Then $J_{[\alpha]} = J_{[\beta]}$.

Lemma 4. ([2, Theorem 22]) *Let R be a Noetherian domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

with $\Delta_\beta = ad - bc \neq 0$. Assume that the following conditions hold:

(1) *c is a unit of R.*

(2) $\Delta_\beta + \varphi_\alpha(c,d)I_{[\alpha]} = R$.

Then $J_{[\alpha]} = J_{[\beta]}$.

Lemma 5. ([2, Theorem 26]) *Let R be a Noetherian domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

with $\Delta_\beta = ad - bc \neq 0$. *Assume that the following conditions hold:*
(1) d *is a unit of* R.
(2) $\Delta_\beta + \varphi_\alpha(-c, -d)I_{[\alpha]} = R$.
Then $J_{[\alpha]} = J_{[\beta]}$.

Since

$$\varphi_\alpha(X, Y) = Y^t + \eta_1 XY^{t-1} + \cdots + \eta_{t-1}X^{t-1}Y + \eta_t X^t,$$

we have $\varphi_\alpha(-a, -b) = (-1)^t\varphi_\alpha(a, b)$ and $\varphi_\alpha(-c, -d) = (-1)^t\varphi_\alpha(c, d)$. Hence $\varphi_\alpha(-a, -b)I_{[\alpha]} = \varphi_\alpha(a, b)I_{[\alpha]}$ and $\varphi_\alpha(-c, -d)I_{[\alpha]} = \varphi_\alpha(c, d)I_{[\alpha]}$.

Theorem 6. *Let* R *be a Noetherian domain and* α *an algebraic element over the quotient field of* R. *Let* β *be a linear fractional transform of* α, *that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

with $\Delta_\beta = ad - bc \neq 0$. *Assume that the following three conditions hold:*
(1) $(a, b, c, d)R = R$.
(2) $\Delta_\beta + \varphi_\alpha(a, b)I_{[\alpha]} = R$.
(3) $\Delta_\beta + \varphi_\alpha(c, d)I_{[\alpha]} = R$.
Then $J_{[\alpha]} = J_{[\beta]}$.

Proof. Let $\mathfrak{p}$ be an arbitrary element of Spec$R$. By the condition (1), we see that $\mathfrak{p} \not\ni a$, $\mathfrak{p} \not\ni b$, $\mathfrak{p} \not\ni c$ or $\mathfrak{p} \not\ni d$. If $\mathfrak{p} \not\ni a$ or $\mathfrak{p} \not\ni b$, then, by Lemmas 2, 3 and condition (2), we have $J_{[\alpha]}R_\mathfrak{p} = J_{R_\mathfrak{p}, [\alpha]} = J_{R_\mathfrak{p}, [\beta]} = J_{[\beta]}R_\mathfrak{p}$. If $\mathfrak{p} \not\ni c$ or $\mathfrak{p} \not\ni d$, then, by Lemmas 4, 5 and condition (3), we have $J_{[\alpha]}R_\mathfrak{p} = J_{R_\mathfrak{p}, [\alpha]} = J_{R_\mathfrak{p}, [\beta]} = J_{[\beta]}R_\mathfrak{p}$. Therefore $J_{[\alpha]} = J_{[\beta]}$. Q.E.D.

Assume that $(a, b, c, d)R = R$. Then the converse of Theorem 6 does not hold, that is, even if $J_{[\alpha]} = J_{[\beta]}$, the conditions $\Delta_\beta + \varphi_\alpha(a, b)I_{[\alpha]} = R$ and $\Delta_\beta + \varphi_\alpha(c, d)I_{[\alpha]} = R$ don't hold in general:

Example 7. *Let* $\mathbf{Z}$ *be the ring of all integers. Set* $R = \mathbf{Z}, \alpha = \sqrt{2}, a = 1, b = 0, c = 0, d = -2$ *and* $\beta = 2/\sqrt{2}\,(= \sqrt{2} = \alpha)$. *Then the following five assertions hold:*
(1) $\Delta_\beta R = 2\mathbf{Z}$.
(2) $I_{[\alpha]} = I_{[\beta]} = R$.
(3) $J_{[\alpha]} = J_{[\beta]} = R$.
(4) $\Delta_\beta + \varphi_\alpha(a, b)I_{[\alpha]} = 2\mathbf{Z} \neq R$.
(5) $\Delta_\beta + \varphi_\alpha(c, d)I_{[\alpha]} = 2\mathbf{Z} \neq R$.

Proof. (1) It is clear from $\Delta_\beta = ad - bc = -2$.
The assertion (2) is obvious from $\varphi_\alpha(X) = \varphi_\beta(X) = X^2 - 2$.
(3) Since $I_{[\alpha]} \subset J_{[\alpha]}$ and $I_{[\beta]} \subset J_{[\beta]}$, we see that $J_{[\alpha]} = J_{[\beta]} = R$ by the assertion (2).

(4) and (5) We obtain $\varphi_\alpha(X,Y) = Y^2 - 2X^2$. Hence $\varphi_\alpha(a,b) = \varphi_\alpha(1,0) = -2$ and $\varphi_\alpha(c,d) = \varphi_\alpha(0,-2) = 4$. Therefore $\Delta_\beta + \varphi_\alpha(a,b)I_{[\alpha]} = (-2,-2)\mathbf{Z} = 2\mathbf{Z} \neq R$ and $\Delta_\beta + \varphi_\alpha(c,d)I_{[\alpha]} = (-2,4)\mathbf{Z} = 2\mathbf{Z} \neq R$.
Q.E.D.

In [2, Proposition 17], we have proved that $J_{[\alpha]} = J_{[a\alpha]}$ if $aR + I_{[\alpha]} = R$. So it would be natural to pose the following question: Assume that the following conditions hold.

(1) $(a,b,c,d)R = R$.

(2) $\Delta_\beta + I_{[\alpha]} = R$.

Then does the equality $J_{[\alpha]} = J_{[\beta]}$ hold? Unfortunately the answer is negative as the following example shows:

Example 8. *Let $R = k[X,Y]$ be a polynomial ring over a field k in two indeterminates X,Y. Set $\alpha = Y/(X-1)$ and $\beta = (Y\alpha+1)/X\alpha$. Then the following assertions hold:*

(1) $(a,b,c,d)R = R$.

(2) $\Delta_\beta = -X$.

(3) $I_{[\alpha]} = (X-1)R$.

(4) $\Delta_\beta + I_{[\alpha]} = R$.

(5) $J_{[\alpha]} = (X-1,Y)R$.

(6) $I_{[\beta]} = XYR$.

(7) $J_{[\beta]} = (XY, Y^2+X-1)R$.

(8) $J_{[\alpha]} \neq J_{[\beta]}$.

Proof. (1) and (2) are clear from $a = X, b = 0, c = Y$ and $d = -1$.

(3) Since $\varphi_\alpha(X) = X - \alpha$, we have $I_{[\alpha]} = \{c \in R; c\alpha \in R\}$. Hence $I_{[\alpha]} = (X-1)R$.

(4) By the assumptions (2) and (3), we know that $\Delta_\beta + I_{[\alpha]} = (-X, X-1)R = R$.

(5) $J_{[\alpha]} = I_{[\alpha]}(1,\eta_1) = (X-1, -(X-1)Y/(X-1)) = (X-1,Y)R$.

(6) Since $\beta = (Y^2+X-1)/XY$, we have $I_{[\beta]} = XYR$.

(7) $J_{[\beta]} = I_{[\beta]}(1, -(Y^2+X-1)/XY) = (XY, Y^2+X-1)R$.

(8) We will prove that $X-1 \notin J_{[\beta]}$. Assume the contrary, that is, $X-1 \in J_{[\beta]} = (XY, Y^2+X-1)R$. Then there exist elements $f(X,Y)$ and $g(X,Y)$ of R such that

$$X - 1 = XYf(X,Y) + (Y^2+X-1)g(X,Y).$$

Substituting X by 0 in the equation above, we have $-1 = (Y^2-1)g(0,Y)$. This is the contradiction. Hence $X-1 \notin J_{[\beta]}$. On the other hand $X-1 \in J_{[\alpha]}$. Therefore $J_{[\alpha]} \neq J_{[\beta]}$. Q.E.D.

Finally we prove a relation between $J_{[\alpha]}$ and $J_{[\beta]}$.

By the proof of [2, Lemma 1], we have the following:

Lemma 9. *Let R be an integral domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

and set $\Delta_\beta = ad - bc$. If $\Delta_\beta \neq 0$, then

$$\Delta_\beta^t I_{[\beta]} \subset \varphi_\alpha(a,b)I_{[\alpha]} \subset I_{[\beta]}$$

where deg $\varphi_\alpha(X) = t$.

Remark 10. The inclusion $I_{[\beta]} \subset \varphi_\alpha(a,b)I_{[\alpha]}$ does not hold in general as Example 7 shows.

Proposition 11. *Let R be an integral domain and α an algebraic element over the quotient field of R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b.c, d \in R,\ a\alpha - b \neq 0)$$

and set $\Delta_\beta = ad - bc$. If $\Delta_\beta \neq 0$, then $\Delta_\beta^t J_{[\beta]} \subset J_{[\alpha]}$.

Proof. Let K be the quotient field of R. Set

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t,\ (\eta_1, \dots, \eta_t \in K)$$

and

$$\varphi_\beta(X) = X^t + \lambda_1 X^{t-1} + \cdots + \lambda_t,\ (\lambda_1, \dots, \lambda_t \in K).$$

By the equality

$$\varphi_\beta(X) = \varphi_\alpha(a,b)^{-1}\varphi_\alpha(aX - c, bX - d),$$

we obtain $\varphi_\alpha(a,b)\lambda_i \in (1, \eta_1, \dots, \eta_t)$. Furthermore, $\varphi_\alpha(a,b) \in (1, \eta_1, \dots, \eta_t)$. Therefore

$$\varphi_\alpha(a,b)(1, \lambda_1, \dots, \lambda_t) \subset (1, \eta_1, \dots, \eta_t).$$

By Lemma 9, we have $\Delta_\beta^t I_{[\beta]} \subset \varphi_\alpha(a,b)I_{[\alpha]}$. Hence

$$\Delta_\beta^t J_{[\beta]} = \Delta_\beta^t I_{[\beta]}(1, \lambda_1, \dots, \lambda_t) \subset \varphi_\alpha(a,b)I_{[\alpha]}(1, \lambda_1, \dots, \lambda_t) \subset I_{[\alpha]}(1, \eta_1, \dots, \eta_t) = J_{[\alpha]}.$$

This shows that $\Delta_\beta^t J_{[\beta]} \subset J_{[\alpha]}$. Q.E.D.

References

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