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Geometrical solution of *P*, *M* and inverse *P*, *M*-problems

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Abstract

The P, M and inverse P, M problems are considered based on the projective geometry. Reciprocity between the direct and inverse problems is put emphasis on. The reciprocity simultaneously solves both problems. P-positions of ordered 4 vectors are introduced. As for the direct problem, a vector p normal to coordinate vectors e_i $(i \in \alpha)$ and column vectors a_i $(i \in \beta)$ of a given matrix A of order n is uniquely determined up to multiple of a positive real number, if and only if α and β are disjoint index-sets of $N = \{1, 2, \cdots, n\}$ and the order of the union of α and β is equal to $n-1$. Among such vectors, the P-position gives rise to two sequences in mutually opposite directions with starting vectors e_i and a_i^*; here, a_i^* is the dual vector of a_i. The sequences provide conditions for the P, M-matrix and the inverse P, M-matrix.
Keywords: P-matrix; M-matrix; Inverse M-matrix.

1. Introduction

A classification of matrices in class Z was primarily published in 1992 by Fiedler and Markham [1]. This classification contains the classes K_0 [2], N_0 [6] and F_0 [8]. Introducing a direct extension to classification of matrices in class Z, the author deals with a geometrical representation of matrices in classes M and inverse M. Here, an inverse M-matrix is a nonsingular M-matrix.

2. Positive position

A matrix A is assumed one in the set of all matrices of order n with real entries, denoted by $\mathcal{M}_n(\mathbf{R})$ with the set of all real numbers $\mathbf{R}$; that is, $A \in \mathcal{M}_n(\mathbf{R})$. Detail of the notation is further described in [3, 4].

Definition 2.1. Let p, q, r, s be vectors. $[p, q, r, s]$ is defined as

$$[p,q,r,s] \overset{\text{def}}{\Leftrightarrow} q, r \in \mathrm{V}(ps),\ q \in \mathrm{V}(pr) - \mathbf{R}^+ r$$

with the polyhedral cone $\mathrm{V}(ps)$ composed of p, s and the set of all positive real numbers $\mathbf{R}^+$. Such vectors p, q, r, s in this order are called to be in positive position, abbreviated to P-position. By taking account of the order of p, q, r, s, they are written as (p, q, r, s), that is, all the P-positions represented by $[p, q, r, s]$ are places in (p, q, r, s).

$[p, q, r, s]$ is equivalently expressed as $q, r \in [p, s]$, $q \in [p, r)$, where $[p, s] \hat{=} \{tp + (1-t)s \mid 0 \le t \le 1\}$, $[p, r) \hat{=} \{tp + (1-t)r \mid 0 < t \le 1\}$. $\{p\} \hat{=} \{q\}$ or $p \hat{=} q$ implies $p = tq$ with some positive real number t, namely,

$\mathbf{R}^+p=\mathbf{R}^+q$; p is called to be positively parallel to q, or p and q are to be positively parallel. $[p,q,r,s]$ is obviously equivalent to $[s,r,q,p]$. If $q=r$, $[p,q\hat{=}r,s]$ stands for $q\in[p,s]$.

The relation $p\sim q$ defined as $p\hat{=}q$ is an equivalent relation. The quotient space of the n-dimensional Euclidian space $\mathbf{R}^n$ by the equivalent relation, $\mathbf{R}^n/_\sim$, is here called a projective space of dimension $n-1$ which is denoted by P^{n-1}. In P^{n-1}, p is identified with q if $p\hat{=}q$.

Proposition 2.2. *Let* $B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$ *such that* $(q\,r)=(p\,s)B$.

0) $q,r\in[p,s]$ $\Leftrightarrow$ $B\geq O,\ b_{1i}+b_{2i}>0\ (i=1,2)$. *1)* $[p,q,r,s]$ $\Leftrightarrow$ $|B|>0,\ B\geq O$.

2) $[p,q\hat{=}r,s]$ $\Leftrightarrow$ $|B|=0,\ B\geq O,\ b_{1i}+b_{2i}>0\ (i=1,2)$. *3)* $[p,r,q,s]$ $\Leftrightarrow$ $|B|<0,\ B\geq O,\ b_{1i}+b_{2i}>0\ (i=1,2)$.

Proof. 0) $q,r\in[p,s]$ gives

$$tq=up+(1-u)s\ (t>0,0\leq u\leq 1),\ wr=(1-v)p+vs\ (w>0,0\leq v\leq 1).$$

Then,

$$B=\begin{pmatrix} \frac{u}{t} & \frac{1-v}{w} \\ \frac{1-u}{t} & \frac{v}{w}\end{pmatrix}\geq O,\ |B|=\frac{u+v-1}{tw},\ b_{11}+b_{21}=\frac{1}{t},\ b_{12}+b_{22}=\frac{1}{w}.$$

Hence, $b_{1i}+b_{2i}>0\ (i=1,2)$.

Conversely, by taking $u=\dfrac{b_{11}}{b_{11}+b_{21}}$, u satisfies $0\leq u\leq 1$. Then, $tq=up+(1-u)s$ with $t=\dfrac{1}{b_{11}+b_{21}}$. Similarly, $v=\dfrac{b_{22}}{b_{12}+b_{22}}$ leads to $wr=(1-v)p+vs$ with $w=\dfrac{1}{b_{12}+b_{22}}$ and $0\leq v\leq 1$. Then, $q,r\in[p,s]$.

1) $q\in[p,r)$ is equivalent to $tq\in[p,wr)$ with some $t,w>0$. Since tq and wr lie on the segment pq, $tq\in[p,wr)$ is equivalent to $1-u<v$ or $1<u+v$. Then, $|B|>0$. By the way B holds, i.e., $B\geq O$ because $[p,q,r,s]$ is $q,r\in[p,s]$.

Conversely, since $|B|>0$ and $B\geq O$, $b_{11}b_{22}>b_{12}b_{21}\geq 0$ and so b_{11}, $b_{22}>0$. Then, $b_{1i}+b_{2i}>0$ $(i=1,2)$. From 0), $q,r\in[p,s]$ follows. By taking $t=\dfrac{1}{b_{11}+b_{21}}$ and $w=\dfrac{1}{b_{12}+b_{22}}$, $|B|>0$ yields $u+v>1$. Hence, $q\in[p,r)$.

2), 3) Proof is quite similar to 1). □

Let α, β be subsets of $N=\{1,2,\cdots,n\}$ and p_β^α be a vector such that $(p_\beta^\alpha,e_i)=0$ $(i\in\alpha)$ and $(p_\beta^\alpha,a_i)=0$ $(i\in\beta)$ with the ith column vector a_i of A. Here, (p,q) is the inner product of two vectors p and q. The order of α is denoted by $|\alpha|$. In the case of $|\alpha|=|\beta|$, $|A_{\alpha,\beta}|$ is the determinant of a matrix $A_{\alpha,\beta}$ composed of entries a_{ij} of A with $i\in\alpha$ and $j\in\beta$. Let σ be a permutation of α defined as $\sigma_\alpha\overset{\text{def}}{=}\begin{pmatrix} i_1\,i_2\cdots i_k \\ 1\,2\cdots k\end{pmatrix}$ with $\alpha=\{i_1,i_2,\cdots,i_k\}$ $(i_1<i_2<\cdots<i_k)$. For simplicity, $|A_{\alpha,\beta}|$ is denoted by $|A_\alpha|$ in case $\alpha=\beta$. If α and β are disjoint, $\alpha\cup\beta$ is written as $\alpha+\beta$. $\{i\}$ is simply written as i. For disjoint α and β such that $|\alpha|+|\beta|=n-1$ and $\{j\}=N-(\alpha+\beta)$, the ith entry of p_β^α is expressed as

$$(p_\beta^\alpha)_i=\begin{cases}(-1)^{\sigma\beta+j(i)+\sigma\beta+j(j)}|A_{\beta+j-i,\beta}| & (i\in\beta+j) \\ 0 & (i\in\alpha)\end{cases}.$$

Especially for $\beta=\varnothing$, $p_\varnothing^{N-i}$ is denoted by p^{N-i}, and $p^{N-i}\hat{=}e_i$ with the ith coordinate vector e_i; for $\alpha=\varnothing$, $p_{N-i}^\varnothing$ or simply p_{N-i} is positively parallel to a_i^* which is the dual vector of the ith column vector a_i of A (see [4]). Now let $\alpha=\{i_1,i_2,\cdots,i_s\}$ $(s\geq 2)$, $\beta=\{j_1,j_2\}$ $\subset\alpha$ $(j_1\neq j_2)$. The ith entry of p is represented as follows.

$$\left(p_{\alpha-\beta}^{N-\alpha+j_1}\right)_i=\begin{cases}(-1)^{\sigma_{\alpha-j_1}(i)+\sigma_{\alpha-j_1}(j_2)}\left|A_{\alpha-(i+j_1),\alpha-\beta}\right| & (i\in\alpha-j_1)\\ 0 & (i\in N-\alpha+j_1)\end{cases},\quad \left(p_{\alpha-\beta}^{N-\alpha+j_2}\right)_i=\begin{cases}(-1)^{\sigma_{\alpha-j_2}(i)+\sigma_{\alpha-j_2}(j_1)}\left|A_{\alpha-(i+j_2),\alpha-\beta}\right| & (i\in\alpha-j_2)\\ 0 & (i\in N-\alpha+j_2)\end{cases},$$

$$\left(p_{\alpha-j_2}^{N-\alpha}\right)_i=\begin{cases}(-1)^{\sigma_\alpha(i)+\sigma_\alpha(j_2)}\left|A_{\alpha-i,\alpha-j_2}\right| & (i\in\alpha)\\ 0 & (i\in N-\alpha)\end{cases},\quad \left(p_{\alpha-j_1}^{N-\alpha}\right)_i=\begin{cases}(-1)^{\sigma_\alpha(i)+\sigma_\alpha(j_1)}\left|A_{\alpha-i,\alpha-j_1}\right| & (i\in\alpha)\\ 0 & (i\in N-\alpha)\end{cases}.$$

In this case, B is given by

$$\left(p_{\alpha-j_2}^{N-\alpha}\,p_{\alpha-j_1}^{N-\alpha}\right)=\left(p_{\alpha-\beta}^{N-\alpha+j_1}\,p_{\alpha-\beta}^{N-\alpha+j_2}\right)B\,. \tag{2.1}$$

Theorem 2.3. *Let* $\left|A_{\alpha-\beta}\right|>0$.

$$\left[p_{\alpha-\beta}^{N-\alpha+j_1},p_{\alpha-j_2}^{N-\alpha},p_{\alpha-j_1}^{N-\alpha},p_{\alpha-\beta}^{N-\alpha+j_2}\right]\iff \left|A_\alpha\right|>0,\ (-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|\ge 0\ (i,j\in\beta).$$

Proof. From the j_1 and j_2th entries of (2.1), it follows that

$$\left|A_{\alpha-\beta}\right|B=\begin{pmatrix}\left|A_{\alpha-j_2}\right| & (-1)^{\sigma_\alpha(j_1)+\sigma_\alpha(j_2)}\left|A_{\alpha-j_2,\alpha-j_1}\right|\\ (-1)^{\sigma_\alpha(j_1)+\sigma_\alpha(j_2)}\left|A_{\alpha-j_1,\alpha-j_2}\right| & \left|A_{\alpha-j_1}\right|\end{pmatrix}. \tag{2.2}$$

In case $\alpha-\beta\ne\varnothing$, Cor. A.2 shows that eqn. (2.1) holds with B in (2.2) for the entries other than the j_1 and j_2th; here, is used

$$\left|A_{\alpha-\beta}\right|\left|A_{\alpha-i,\alpha-j_2}\right|=(-1)^{[i/j_1]+[j_2/j_1]}\left|A_{\alpha-(i+j_1),\alpha-\beta}\right|\left|A_{\alpha-j_2}\right|+(-1)^{[i/j_2]+[j_1/j_2]}\left|A_{\alpha-(i+j_2),\alpha-\beta}\right|\left|A_{\alpha-j_1,\alpha-j_2}\right|.$$

If $i\notin\alpha$, the ith equation of (2.1) becomes $0=0b_{1i}+0b_{2i}$. By Prop. 2.2.1), $\left[p_{\alpha-\beta}^{N-\alpha+j_1},p_{\alpha-j_2}^{N-\alpha},p_{\alpha-j_1}^{N-\alpha},p_{\alpha-\beta}^{N-\alpha+j_2}\right]$ is equivalent to

$$\begin{Vmatrix}\left|A_{\alpha-j_2}\right| & \left|A_{\alpha-j_2,\alpha-j_1}\right|\\ \left|A_{\alpha-j_1,\alpha-j_2}\right| & \left|A_{\alpha-j_1}\right|\end{Vmatrix}>0,\ (-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|\ge 0\ (i,j\in\beta).$$

Combined with Prop. A.4, the equivalence of the theorem is seen. □

From Prop. 2.2, the following properties are immediately obtained.

Corollary 2.4. *Let* $\left|A_{\alpha-\beta}\right|>0$.

1) $\left[p_{\alpha-\beta}^{N-\alpha+j_1},p_{\alpha-j_2}^{N-\alpha}=p_{\alpha-j_1}^{N-\alpha},p_{\alpha-\beta}^{N-\alpha+j_2}\right]\iff \left|A_\alpha\right|=0,(-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|\ge 0\ (i,j\in\beta),$
$\left|A_{\alpha-i}\right|+(-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|>0\ (i,j\in\beta,i\ne j).$

2) $\left[p_{\alpha-\beta}^{N-\alpha+j_1},p_{\alpha-j_1}^{N-\alpha},p_{\alpha-j_2}^{N-\alpha},p_{\alpha-\beta}^{N-\alpha+j_2}\right]\iff \left|A_\alpha\right|<0,\ (-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|\ge 0\ (i,j\in\beta),$
$\left|A_{\alpha-i}\right|+(-1)^{\sigma_\alpha(i)+\sigma_\alpha(j)}\left|A_{\alpha-i,\alpha-j}\right|>0\ (i,j\in\beta,i\ne j).$

3. Geometrical representations

For given α and β such that $\left[p_{\alpha-\beta}^{N-\alpha+j2}, p_{\alpha-j1}^{N-\alpha}, p_{\alpha-j2}^{N-\alpha}, p_{\alpha-\beta}^{N-\alpha+j1}\right]$, each component in the brackets is uniquely determined except for positive coefficient. In fact, since $\beta=\{j_1,j_2\}\subset\alpha$, the union of the super and sub index-sets of p contains at most $N-j_1$ for the first two vectors of the brackets and $N-j_2$ for the rest two. f and g are defined as $p_{\alpha-j1}^{N-\alpha}=f\left(p_{\alpha-\beta}^{N-\alpha+j2}\right)$ and $p_{\alpha-\beta}^{N-\alpha+j2}=g\left(p_{\alpha-j1}^{N-\alpha}\right)$; then, $g\circ f=id$. Consider the following two sequences from the left to the right (L-sequence) and in the reverse direction (R-sequence) :

$$e_{j1}\hat{=}p^{N-j1}\underset{g_{n-1}}{\overset{f1}{\leftrightarrow}}p_{j2}^{N-(j1+j2)}\underset{g_{n-2}}{\overset{f2}{\leftrightarrow}}p_{j2+j3}^{N-(j1+j2+j3)}\underset{g_{n-3}}{\overset{f3}{\leftrightarrow}}\cdots\underset{g1}{\overset{f_{n-1}}{\leftrightarrow}}p_{j2+j3+\cdots+jn}^{N-(j1+j2+\cdots+jn)}=p_{N-j1}\hat{=}a_{j1}^{\bullet}.$$

In general, for given disjoint index-sets $\alpha=\{i_1,i_2,\cdots,i_s\}$ and $\beta=\{j_1,j_2,\cdots,j_t\}$ with $\alpha+\beta=N-i$, the L-sequence:

$$e_i\hat{=}p^{N-i}\overset{f1}{\mapsto}p_{j1}^{N-(i+j1)}\overset{f2}{\mapsto}p_{j1+j2}^{N-(i+j1+j2)}\overset{f3}{\mapsto}\cdots\overset{ft}{\mapsto}p_{j1+j2+\cdots+jt}^{N-(i+j1+j2+\cdots+jt)}=p_{\beta}^{\alpha}$$

yields a mapping f_β such that $p_\beta^\alpha=f_\beta(e_i)$ with $f_\beta=f_t\circ f_{t-1}\circ\cdots\circ f_1$. On the other hand, the R-sequence:

$$a_i^{\bullet}\hat{=}p_{N-i}\overset{g1}{\mapsto}p_{N-(i+i1)}^{i1}\overset{g2}{\mapsto}p_{N-(i+i1+i2)}^{i1+i2}\overset{g3}{\mapsto}\cdots\overset{g_s}{\mapsto}p_{N-(i+i1+i2+\cdots i_s)}^{i1+i2+\cdots i_s}=p_\beta^\alpha$$

provides g_α defined as $p_\beta^\alpha=g_\alpha\left(a_i^\bullet\right)$ with $g_\alpha=g_s\circ g_{s-1}\circ\cdots\circ g_1$.

Let

$$L_i(t)\overset{\text{def}}{=}\left\{p_\beta^{N-(i+\beta)}\,\middle|\,\exists f_\beta;e_i\mapsto p_\beta^{N-(i+\beta)},|\beta|=t\right\},\ L_i(0)\overset{\text{def}}{=}\left\{p^{N-i}\right\}\hat{=}\{e_i\}$$

and

$$R_i(s)\overset{\text{def}}{=}\left\{p_{N-(i+\alpha)}^{\alpha}\,\middle|\,\exists g_\alpha;a_i^\bullet\mapsto p_{N-(i+\alpha)}^{\alpha},|\alpha|=s\right\},\ R_i(0)\overset{\text{def}}{=}\{p_{N-i}\}\hat{=}\{a_i^\bullet\}.$$

Then,

$$L_i(0)=\left\{p^{N-i}\right\}\hat{=}\{e_i\},$$

$$L_i(1)=\left\{p_1^{N-(i+1)},p_2^{N-(i+2)},\cdots,p_{i-1}^{N-(i+(i-1))},p_{i+1}^{N-(i+(i+1))},\cdots,p_n^{N-(i+n)}\right\},$$

$$L_i(2)=\left\{p_{1+2}^{N-(i+1+2)},p_{1+3}^{N-(i+1+3)},\cdots,p_{1+(i-1)}^{N-(i+1+(i-1))},p_{1+(i+1)}^{N-(i+1+(i+1))},\cdots,p_{1+n}^{N-(i+1+n)},p_{2+3}^{N-(i+2+3)},\cdots,p_{(n-1)+n}^{N-(i+(n-1)+n)}\right\}$$

$$\vdots$$

$$L_i(n-1)=\{p_{N-i}\}\hat{=}\{a_i^\bullet\},$$

and reciprocally

$$R_i(0)=\{p_{N-1}\}\hat{=}\{a_i^\bullet\},$$

$$R_i(1)=\left\{p_{N-(i+1)}^{1},p_{N-(i+2)}^{2},\cdots,p_{N-(i+(i-1))}^{i-1},p_{N-(i+(i+1))}^{i+1},\cdots,p_{N-(i+n)}^{n}\right\},$$

$$R_i(2)=\left\{p_{N-(i+1+2)}^{1+2},p_{N-(i+1+3)}^{1+3},\cdots,p_{N-(i+1+(i-1))}^{1+(i-1)},p_{N-(i+1+(i+1))}^{1+(i+1)},\cdots,p_{N-(i+1+n)}^{1+n},p_{N-(i+2+3)}^{2+3},\cdots,p_{N-(i+(n-1)+n)}^{(n-1)+n}\right\}$$

$$\vdots$$

$$R_i(n-1)=\left\{p^{N-i}\right\}\hat{=}\{e_i\}.$$

Thus, the following propositions are derived.

Proposition 3.1.

$$p_\beta^\alpha \in L_i(k)\ (0 \le \exists k \le n-1) \ \Rightarrow\ p_\beta^\alpha \in \mathrm{V}I.$$

Here, $\mathrm{V}I$ *is the cone generated by the coordinate vectors* e_i $(0 \le i \le n)$.

Theorem 3.2.

Once there exists k *such that* $L_i(k) = R_i(n-1-k)$ *and* $|L_i(k)| = \binom{n-1}{k}$ *with the binomial coefficient* $\binom{n-1}{k}$, $L_i(j) = R_i(n-1-j)$ *with* $|L_i(j)| = \binom{n-1}{j}$ *is satisfied for any* j *such that* $0 \le j \le n-1$*; especially,* $L_i(n-1) \mathrel{\hat=} \{a_i^\bullet\}$, $R_i(n-1) \mathrel{\hat=} \{e_i\}$ *hold.*

$L_i(k)$ gives $|A_{i+\beta}| > 0$ $(\beta \subset N-i, \beta = \{j_r \mid j_1 \le j_r < j_{r+1} \le j_k\})$ and $R_i(k+1)$ yields $|A_{N-\alpha}| > 0$ $(\alpha \subset N-i, \alpha = \{i_r \mid i_1 \le i_r < i_{r+1} \le i_k\})$. Therefore,

Theorem 3.3. *Either* $R_i(n-1) = L_i(0) \mathrel{\hat=} \{e_i\}$ *or* $L_i(n-1) = R_i(0) \mathrel{\hat=} \{a_i^\bullet\}$ *for any* i *in* N *is equivalent to* $A \in \mathcal{M}$. *Here,* $\mathcal{M}$ *denotes the set of all M-matrices.*

Combined with Theorem 3.3,

Corollary 3.4. *For any* i *in* N*, there exists* k $(0 \le k \le n-1)$ *such that* $L_i(k) = R_i(n-1-k)$, $|L_i(k)| = \binom{n-1}{k}$, *if and only if* $A \in \mathcal{M}$.

The sequence of $\{L_i(k)\}_{0 \le k \le s_i}$ relates to the P, M-problem of A whether A is a P, M-matrix or not, while the sequence of $\{R_i(k)\}_{0 \le k \le s_i}$ to the inverse P, M-problem, by substituting for $a_i^\bullet$, where a_i is a given matrix A with all nonnegative entries. In the P, M-problem, $L_i(1) = \binom{n-1}{1}$ $(\forall i \in N)$ implies that A is an L-matrix. In fact, $\left(p^{N-i}, p_j^{N-(i+j)}, p_i^{N-(i+j)}, p^{N-j}\right)$ are in the P-position, so that $0 < \left(p_j^{N-(i+j)}\right)_j = a_{jj}$ and $0 \le \left(p_j^{N-(i+j)}\right)_i = -a_{ij}$, $\left(p_j^{N-(i+j)}\right)_r = 0\ (r \in N-(i+j))$. Therefore, the P-problem is equivalent to the M-problem.

In the inverse problem, A is à priori assumed nonsingular. It is further necessary for A^{-1} to be an M-matrix that A^{-1} is an L-matrix and $|A^{-1}| > 0$ or $|A| > 0$. $A^{-1} = \frac{{}^t A^\bullet}{|A|}$ yields $\left(A^{-1}\right)^\bullet = \frac{{}^t A}{|A|}$. Here, $A^\bullet$ is the dual matrix of A and ${}^t A$ denotes the transpose of A. For L-matrix A^{-1} with $|A| > 0$, A^{-1} is an M-matrix, if and only if $\mathrm{V}(\mathrm{A}^{-1})^\bullet \subset \mathrm{V}I$ (Theorem 3.6 of [4]). Thus, for the inverse problem we consider a matrix with all nonnegative entries. The reciprocity of the R-sequence with the L-sequence leads to the following theorem and corollary, corresponding to Theorem 3.3 and Cor. 3.4.

Theorem 3.5. *Let* $A^{-1} = (a'_{ij})$.

Either $R_i(n-1) = L_i(0) \mathrel{\hat=} \{e_i\}$ *or* $L_i(n-1) = R_i(0) \mathrel{\hat=} \{a'^\bullet_i\}$ $(\forall i \in N)$ $\Leftrightarrow$ $A^{-1} \in \mathcal{M}$.

Corollary 3.6. *Let* $A^{-1} = (a'_{ij})$.

$$\forall i \in N, \exists k; 0 \le k \le n-1, L_i(k) = R_i(n-1-k), |L_i(k)| = \binom{n-1}{k} \ \Leftrightarrow\ A^{-1} \in \mathcal{M}.$$

Here, it is noted that $R_i(n-1) \hat{=} \{e_i\}$ for any i in N implies $A^{-1} \in \mathcal{L}$ with the set of all L-matrices, $\mathcal{L}$. In fact, as mensioned above in the P, M-problem the offdiagonal (i,j)-entry of A^{-1} is given by $-\left(p_j^{N-(i+j)}\right)_i$ which is nonpositive and $\left(A^{-1}\right)_{jj} = \left(p_j^{N-(i+j)}\right)_j > 0$, since $\left(e_i, p_j^{N-(i+j)}, p_i^{N-(i+j)}, e_j\right)$ are in the P-position.

Example 3.7.

Let $N = \{1,2,3,4,5\}$ and $i = 1$. By omitting + in the index, all elements of $L_1(k)$ are ėxpressed in case $|L_1(k)| = \binom{n-1}{k}$.

$$L_1(0) = \{p^{2345}\} \hat{=} \{e_1\},$$
$$L_1(1) = \{p_2^{345}, p_3^{245}, p_4^{235}, p_5^{234}\};$$

thus, $|A_{12}|$, $|A_{13}|$, $|A_{14}|$, $|A_{15}| > 0$ corresponding to the vectors in the braces of $L_1(1)$ (Theorem 2.3),

$$L_1(2) = \{p_{23}^{45}, p_{24}^{35}, p_{25}^{34}, p_{34}^{25}, p_{35}^{24}, p_{45}^{23}\};$$

thus, $|A_{123}|$, $|A_{124}|$, $|A_{125}|$, $|A_{134}|$, $|A_{135}|$, $|A_{145}| > 0$ corresponding to the vectors of $L_1(2)$,

$$L_1(3) = \{p_{234}^{5}, p_{235}^{4}, p_{245}^{3}, p_{345}^{2}\};$$

thus, $|A_{1234}|$, $|A_{1235}|$, $|A_{1245}|$, $|A_{1345}| > 0$ corresponding to the vectors of $L_1(3)$,

$$L_1(4) = \{p_{2345}\} \hat{=} \{a_1^{\bullet}\};$$

thus, $|A| = |A_{12345}| > 0$. The above procedure is schematically drawn in Fig. 3.1.

Reciprocally, for the R-sequence, $\binom{n-1}{(n-1)-k} = \binom{n-1}{k}$ is the maximum order of $R_1(k)$. In this case, all elements of $R_1(k)$ are given by

$$R_1(0) = \{p_{2345}\} \hat{=} \{a_1^{\bullet}\},$$
$$R_1(1) = \{p_{345}^{2}, p_{245}^{3}, p_{235}^{4}, p_{234}^{5}\};$$

thus, $|A| = |A_{12345}| > 0$,

$$R_1(2) = \{p_{45}^{23}, p_{35}^{24}, p_{34}^{25}, p_{25}^{34}, p_{24}^{35}, p_{23}^{45}\};$$

thus, $|A_{1345}|$, $|A_{1245}|$, $|A_{1235}|$, $|A_{1234}| > 0$ corresponding to the vectors in the braces of $R_1(1)$,

$$R_1(3) = \{p_5^{234}, p_4^{235}, p_3^{245}, p_2^{345}\};$$

thus, $|A_{145}|$, $|A_{135}|$, $|A_{134}|$, $|A_{125}|$, $|A_{124}|$, $|A_{123}| > 0$ corresponding to the vectors of $R_1(2)$,

$$R_1(4) = \{p^{2345}\} \hat{=} \{e_1\};$$

thus, $|A_{15}|$, $|A_{14}|$, $|A_{13}|$, $|A_{12}| > 0$ corresponding to the vectors of $R_1(3)$.

$L_1(4)=\{p_{2345}\}\mathrel{\hat=}\{a_1^*\}$ assures that $\left[p_{\alpha-\beta}^{N-\alpha+j1}, p_{\alpha-j2}^{N-\alpha}, p_{\alpha-j1}^{N-\alpha}, p_{\alpha-\beta}^{N-\alpha+j2}\right]$ holds for all α and β with $|\alpha|=2,3,4,5$, $\beta\subset\alpha$ and $|\beta|=2$. Thus, all principal minors of an L-matrix A with the indices containing 1 are positive. Furthermore, $L_i(4)\mathrel{\hat=}\{a_i^*\}$ $(i=2,3,4,5)$ is asserted; hence, A is a P-matrix and so an M-matrix.

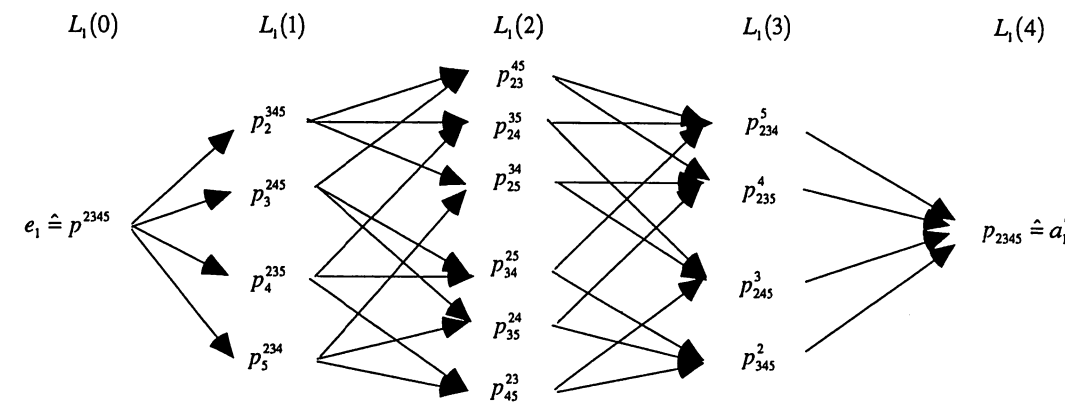

Fig. 3.1 Schematic daiagram of the L-sequence for $N=\{1,2,3,4,5\}$.

4. *P*, *M* and inverse *M*-problem in low dimensional cases

4.1 *P* and *M*-problem

The P and M-problem on an L-matrix A of orders 3 and 4 is discussed. For an L-matrix, the P-problem is equivalent to the M-problem. Figure 4.1 to 4.3 are of order 3 and 4.4 of order 4. Figure 4.1 is an example that $L_1(1)=\{p_3^2\}$. $p_2^3\notin L_1(1)$, since $p_{12}\mathrel{\hat=}a_3^*\notin[p_2^3,p_2^1]$. $p_2^3\notin L_1(1)$ implies $|A_{12}|\le 0$. The figure indicates a case of $|A_{12}|\le 0$. Then, A is neither a P nor an M-matrix. The second one shown in Fig. 4.2 is not a P or an M-matrix, either. $L_i(1)$ $(i=1,2,3)$ is of order $\binom{2}{1}$, while $L_i(2)=\varnothing$ for $i=1,2,3$. In fact, $[p_2^3,p_{23},p_{12},p_2^1]$ is out of the P-position; then, $L_1(2)=\varnothing$. Similarly, $L_2(2), L_3(2)=\varnothing$ are seen by $(p_1^3,p_{13},p_{12},p_1^2)$ and $(p_2^1,p_{12},p_{23},p_2^3)$ out of the P-position, respectively. However, $[p_{i+1}^i,p_{i+1i+2},p_{ii+1},p_{i+1}^{i+2}]$ with the indices represented by 1, 2, 3 modulo 3 holds for $i=1,2,3$. It implies that $|A|<0$. From $|L_i(1)|=2$ for $i\in N$, all principal minors of order 2 are positive. Figure 4.2, therefore, shows a projective geometrical representation of the almost P-matrix of order 3 (see [7]). Figure 4.3 depicts the case where all linear arrays of 4 vectors are in the P-position, so that A is a P or an M-matrix. Figure 4.4 exhibits a case of a P, M-matrix of order 4.

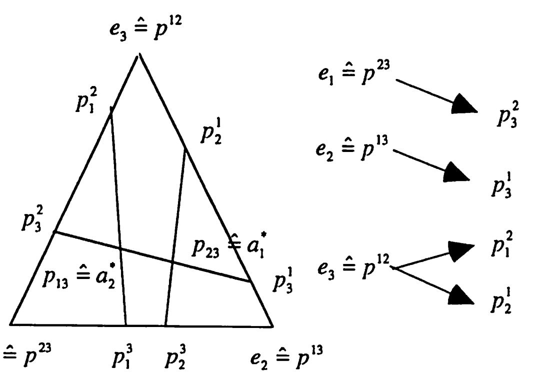

Fig. 4.1 Vectors p_β^α for a non M-matrix A of order 3 in the projective space P^2.

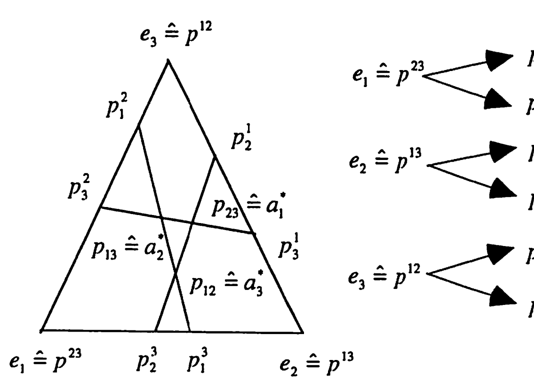

Fig. 4.2 Vectors p_β^α for a non M-matrix A of order 3 in the projective space P^2 (almost P-matrix [7]).

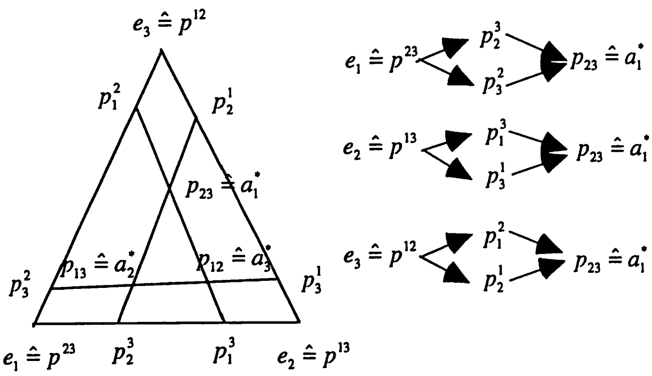

Fig. 4.3 Vectors p_β^α for an M-matrix A of order 3 in the projective space P^2.

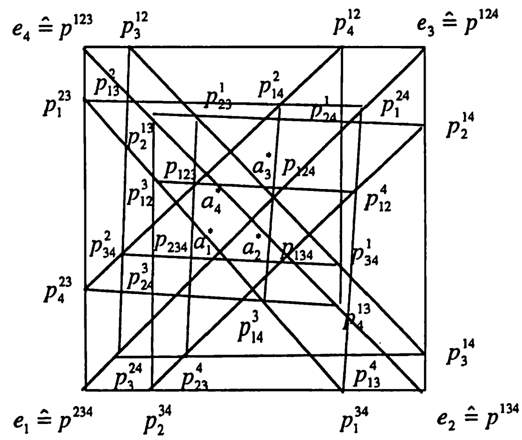

Fig. 4.4 Vectors p_β^α in the projective space P^3.

4.2 Completion of inverse M-matrix in projective space

Consider geometrically the inverse M-problem of a matrix of order 3 in the 2-dimensional projective space P^2. Let $A = (a_{ij})$ be a nonsingular nonnegative matrix of order 3. a_1 and a_2 are plotted in the cone VI (Fig. 4.5). We determine the domain of a_3, $D(a_3)$, such that A is an inverse M-matrix. The condition of the inverse M-matrix requires that $[p_3^2, a_1, a_2, p_3^1]$ is in the P-position. Furthermore, $p_3^2 \in [e_1, e_3)$ and $p_3^1 \in [e_2, e_3)$ are prerequisite for the existence of p_1^2 and p_2^1 in the P-positions of $[e_3, p_1^2, p_3^2, e_1]$ and $[e_2, p_3^1, p_2^1, e_3]$, respectively. By the condition for the existence of p_2^1 so as to satisfy $[e_2, p_3^1, p_2^1, e_3]$, a_3 should be found in $\mathrm{V}(a_1 p_3^1 e_3)$. The condition for p_1^2 requires $a_3 \in \mathrm{V}(a_2 e_3 p_3^2)$. The nonsingularity of A prohibits $a_3 \in [p_3^2, p_3^1] = \mathrm{V}(p_3^2 p_3^1)$. Hence, $a_3 \in \mathrm{V}(a_1 a_2 e_3) - \mathrm{V}(a_1 a_2)$. Further, the condition that $[e_1, p_2^3, p_1^3, e_2]$ is in the P-position restricts the domain $D(a_3)$ to $\mathrm{V}(e_3 q_1 q_2 q_3)$. Without $a_i \doteq e_i$ $(i = 1,2)$, the three vectors q_i are given by $\{q_1\} \doteq \mathrm{V}(e_3 a_1) \cap \pi(e_2 a_2)$, $\{q_2\} \doteq \pi(e_1 a_1) \cap \pi(e_2 a_2) \cap \mathrm{V}I$, $\{q_3\} \doteq \mathrm{V}(e_3 a_2) \cap \pi(e_1 a_1)$; here, $\pi(ab)$ is the plane generated by linearly independent vectors a and b. In case $a_1 \doteq e_1$ or $a_2 \doteq e_2$, the domain $D(a_3)$ is $\mathrm{V}(e_3 q_0 a_2) - \mathrm{V}(a_2)$ or $\mathrm{V}(e_3 a_1 q_4) - \mathrm{V}(a_1)$, respectively, where $\{q_0\} \doteq \mathrm{V}(e_3 e_1) \cap \pi(e_2 a_2)$ and $\{q_4\} \doteq \mathrm{V}(e_3 e_2) \cap \pi(e_1 a_1)$. Conversely, geometrical consideration readily admits $L_i(2) = \{a_i\}$ $(i = 1,2,3)$ (Fig. 4.6). Thus, A is an inverse M-matrix.

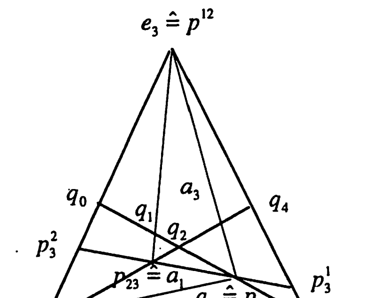

Fig. 4.5 Projective geometrical representation of the inverse M-matrix of order 3.

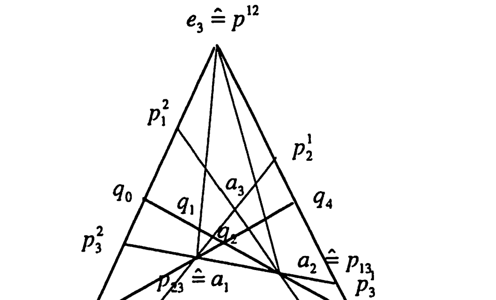

Fig. 4.6 Construction of p_β^α so as to be in the P-position based on a_1, a_2 and a_3.

5. Concluding remarks

The M and inverse M-problems of an L-matrix correspond to the L and R-sequences, respectively. According to the reciprocity of the two sequences, it suffices to treat one of them. For both the M and inverse M-problems, solution is readily found in the projective space of the cone encased by the coordinate vectors.

References

[1] M. Fiedler and T. L. Markham, A classification of matrices of class Z, *Linear Algebra Appl.* 173: 115-124 (1992).

[2] M. Fiedler and V. Pták, On matrices with nonpositive off-diagonal elements and positive principal minors, *Czechoslovak Math. J.*, **12**, 382-400 (1962).

[3] Y. Iwasaki, Heritage of original matrix from its preconditioner in the convergent splitting, *RIMS Report*, 1084: 87-102 (1999).

[4] Y. Iwasaki, Geometrical aspect of generalized strictly diagonal dominance, *Far East J, Math. Sci. (FJMS)* **5**(2): 135-154 (2002).

[5] Y. Iwasaki, Idendity expression of determimant (to be submitted).

[6] G. Johnson, A generalization of N-matrices, *Linear Algebra Appl.*, 48: 201-217 (1982).

[7] T. Parthasarathy and G. Ravindran, N-matrices, *Linear Algebra Appl.* 139: 89-102 (1990).

[8] R. L. Smith, On the spectrum of N_0-matrices, *Linear Algebra Appl.*, 83: 129-134 (1986).

Appendix

Let ν, β be nonempty subsets of α, and $\nu \cap \beta = \varnothing$ with $\nu = \{1, 2, \cdots, |\nu|\}$.

Theorem A.1. [5]

$$\sum_{\substack{\gamma \subset \nu^{C} \\ |\gamma|=|\beta|}} |A_{\nu+\gamma, \nu+\beta}| |\overline{A}_{\gamma^{C}, \beta^{C}}| = |A_\nu| |A_\alpha|,$$

or

$$\sum_{\substack{\gamma \cap \varepsilon = \varnothing \\ \gamma+\varepsilon=\nu^{C} \\ |\gamma|=|\beta|}} |A_{\nu+\gamma, \nu+\beta}| |\overline{A}_{\nu+\varepsilon, \nu+\delta}| = \sum_{\substack{\gamma \cap \varepsilon = \varnothing \\ \gamma+\varepsilon=\nu^{C} \\ |\gamma|=|\beta|}} |A_{\gamma^{C}, \beta^{C}}| |\overline{A}_{\varepsilon^{C}, \delta^{C}}| = \begin{cases} |A_\nu| |A_\alpha| & (\beta \cap \delta = \varnothing) \\ 0 & (\beta \cap \delta \neq \varnothing) \end{cases}$$

for a subset δ of α and $\delta \subset \nu^{C}$. If $\beta \cap \delta = \varnothing$, then $|\nu| + |\beta| + |\delta| = |\alpha|$.

By remarking the relation $\sigma(i) = \sigma_{\alpha-j}(i) + [i/j]$ for $i \in \alpha - j$ with $[i/j] \overset{\text{def}}{=} \begin{cases} 0 & (i < j) \\ 1 & (i > j) \end{cases}$, the following corollaries are obtained as special cases of Theorem A.1. For Cor. A.2 to Prop. A.4, one defines $\beta = \{j_1, j_2\} \subset \alpha$ $(j_1 \neq j_2)$.

Corollary A.2. *Let $i \in \alpha - \beta$.*

$$|A_{\alpha-\beta}| |A_{\alpha-i, \alpha-j2}| - (-1)^{[i/j1]+[j2/j1]} |A_{\alpha-(i+j1), \alpha-\beta}| |A_{\alpha-j2}| - (-1)^{[i/j2]+[j1/j2]} |A_{\alpha-(i+j2), \alpha-\beta}| |A_{\alpha-j1, \alpha-j2}| = 0.$$

Corollary A.3. *Let* $k,l \in \alpha-\beta$ $(k \neq l)$.

$$\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\left|A_{\alpha-\beta,\alpha-(l+j1)}\right| - (-1)^{[k/j1]+[k/j2]+[l/j1]+[l/j2]}\left|A_{\alpha-\beta,\alpha-(l+j2)}\right|\left|A_{\alpha-\beta,\alpha-(k+j1)}\right|$$
$$+(-1)^{[k/j1]+[k/j2]+[l/j1]+[l/j2]}\left|A_{\alpha-\beta}\right|\left|A_{\alpha-\beta,\alpha-(k+l)}\right| = 0.$$

The following proposition A.4 is essential to attain Theorem 2.3. Though Cor. A.3 is a special case, the proof is given below reflecting 3 parts of Theorem A.1 as k and l in Theorem 4.1 [5].

Proposition A.4. *For* $s \geq 2$ $(s=|\alpha|)$,

$$\begin{vmatrix} \left|A_{\alpha-j2}\right| & \left|A_{\alpha-j2,\alpha-j1}\right| \\ \left|A_{\alpha-j1,\alpha-j2}\right| & \left|A_{\alpha-j1}\right| \end{vmatrix} = \left|A_{\alpha-\beta}\right|\left|A_{\alpha}\right| \ \textit{with} \ \left|A_{\emptyset}\right| \stackrel{\text{def}}{=} 1.$$

Proof. Let $\mu_{tu}(k) = j_t - [j_t/j_u] + k - [k/j_u]$, $\nu_{tu}(k) = j_t - [j_t/j_u] + k - [k/j_t]$ $(t,u \in \{1,2\})$.

$$\left|A_{\alpha-j2}\right|\left|A_{\alpha-j1}\right| - \left|A_{\alpha-j1,\alpha-j2}\right|\left|A_{\alpha-j2,\alpha-j1}\right|$$

$$= \left(\sum_{k\in\alpha-j2}(-1)^{\mu12(k)}a_{j1k}\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\right)\left(\sum_{l\in\alpha-j1}(-1)^{\mu21(l)}a_{j2l}\left|A_{\alpha-\beta,\alpha-(l+j1)}\right|\right)$$

$$-\left(\sum_{k\in\alpha-j2}(-1)^{\nu21(k)}a_{j2k}\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\right)\left(\sum_{l\in\alpha-j1}(-1)^{\nu12(l)}a_{j1l}\left|A_{\alpha-\beta,\alpha-(l+j1)}\right|\right)$$

$$= \sum_{\substack{k\in\alpha-j2\\ l\in\alpha-j1}}(-1)^{\mu12(k)+\mu21(l)}\left(a_{j1k}a_{j2l} - a_{j2k}a_{j1l}\right)\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\left|A_{\alpha-\beta,\alpha-(l+j1)}\right|$$

$$= \sum_{\substack{k=j1\\ l\in\alpha-j1}}(-1)^{\mu21(l)}\left|A_{\beta,l+j1}\right|\left|A_{\alpha-\beta}\right|\left|A_{\alpha-\beta,\alpha-(l+j1)}\right| + \sum_{\substack{j1\neq k\in\alpha-j2\\ l=j2}}(-1)^{\mu12(k)}\left|A_{\beta,(k+j2)}\right|\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\left|A_{\alpha-\beta}\right|$$

$$+\sum_{\substack{k,l\in\alpha-\beta\\ k<l}}(-1)^{\mu12(k)+\mu21(l)}\left|A_{\beta,k+l}\right|\left(\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\left|A_{\alpha-\beta,\alpha-(l+j1)}\right| - (-1)^{[k/j1]+[k/j2]+[l/j1]+[l/j2]}\left|A_{\alpha-\beta,\alpha-(l+j2)}\right|\left|A_{\alpha-\beta,\alpha-(k+j1)}\right|\right). \quad \text{(A.1)}$$

By Cor. A.3, the last end side of (A.1) is shown as

$$\left|A_{\alpha-\beta,\alpha-(k+j2)}\right|\left|A_{\alpha-\beta,\alpha-(l+j1)}\right| - (-1)^{[k/j1]+[k/j2]+[l/j1]+[l/j2]}\left|A_{\alpha-\beta,\alpha-(l+j2)}\right|\left|A_{\alpha-\beta,\alpha-(k+j1)}\right|$$

$$= -(-1)^{[k/j1]+[k/j2]+[l/j1]+[l/j2]}\left|A_{\alpha-\beta}\right|\left|A_{\alpha-\beta,\alpha-(k+l)}\right|.$$

Theorem A.1 is, therefore, seen

$$(\text{the last side of (A.1)}) = \left|A_{\alpha-\beta}\right|\left|A_{\alpha}\right|$$

by the division of the 3 parts according to $k=j_1, l\in\alpha-j_1$; $j_1\neq k\in\alpha-j_2, l=j_2$; $k,l\in\alpha-\beta, k<l$. □