Remaks on the Flatness of Anti-Integral Extensions

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Let R be a Noetherian integral domain with quotient field K and let $R[X]$ be a polynomial ring over R. Let α be an element of an algebraic field extension L of K and let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism sending X to α. Let $\varphi_\alpha(X)$ denote the monic minimal polynomial of α over K with $\deg(\varphi_\alpha(X)) = d$ and write

$$\varphi_\alpha(X) := X^d + \eta_1 X^{d-1} + \cdots + \eta_d,$$

where $\eta_1, \ldots, \eta_d \in K$. Let

$$I_{[\alpha]} := \bigcap_{i=1}^{d} (R :_R \eta_i),$$

where $(R :_R \eta_i) := \{ c \in R \mid c\eta_i \in R \}$. Then $I_{[\alpha]}$ is an ideal of R. The ideal $I_{[\alpha]}$ is called the *generalized denominator ideal* of α. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]}C(\varphi_\alpha(X))$,which is an ideal of R and contains $I_{[\alpha]}$. An element $\alpha \in L$ is called an *anti-integral* element of degree d over R if $Ker(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$ (cf.[1]). When α is an anti-integral element over R, $R[\alpha]$ is called an *anti-integral extension* of R. Any unexplained terminology or notation is standard,as in [2].

Let B be a subring of an integral domain A. Put $I_{B,[\alpha]} := \bigcap_{i=1}^{d}(B :_B \eta_i)$ and $J_{B,[\alpha]} := I_{B,[\alpha]} C(\varphi_\alpha(X))$. Then we obtain the following:

Theorem 1. *Let R be an integral domain with quotient field K and let α be an algebraic element over K. Let B be an intermediate ring between R and $A = R[\alpha]$. Suppose that α is an anti-integral element over B. Put $\beta = g(\alpha) \in A$ for some polynomial $g(X) \in R[X]$. If $C(g(X) - \beta) = R$, then A is flat over B.*

Proof. Since $A = R[\alpha]$, then there exists an B-algebra homomorphism $\tilde{\pi} : B[X] \to A$ sending X to α. Then we have $g(X) - \beta \in Ker(\tilde{\pi})$. Since α is anti-integral over B, we see $C(Ker(\tilde{\pi})) = J_{B,[\alpha]}$. It follows that $C(g(X) - g(\alpha)) \subseteq J_{B,[\alpha]}$, and thus $J_{B,[\alpha]} = B$ by assumption. Therefore A is flat over B (cf.[2,Proposition3.4]). □

Let $A \subset B$ be an extension of integral domains. Recall that $tr.deg_A B$ denotes the transcendence degree of the quotient field of B over that of A. For a prime ideal p in $Spec(A)$, let $\kappa(p) = A_p/pA_p$ be a residue field at p.

Theorem 2. *Let R be an integral domain with quotient field K and let α be an algebraic element over K. Let β be an element of $A = R[\alpha]$ such that $\beta = g(\alpha)$ for some monic polynomial $g(X) \in R[X]$. Let $B = R[\beta]$ and assume that β is anti-integral over R. Then the following statements hold:*

(1) $\{p \in Spec(R) | A_p$ *is flat over* $R_p\} = \{p \in Spec(R) | B_p$ *is flat over* $R_p\}$;
(2) $Im(Spec(A) \to Spec(R)) = Im(Spec(B) \to Spec(R))$.

Proof.(1)Take a prime ideal p in Spec(R).

$(\supseteq)$: Suppose that A_p is not flat over R_p. Then $P = pA$ is a prime ideal of A such that $p = P \cap R$ and $tr.deg_{\kappa(p)}\kappa(P) = 1$ (cf.[2.Proposition2.6]). Put $\wp = P \cap B$. Since $A = R[\alpha]$ and $g(\alpha) - \beta = 0$, it follows that A is integral over B, and hence $tr.deg_{\kappa(\wp)}\kappa(P) = 0$. Therefore we get $tr.deg_{\kappa(p)}\kappa(\wp) = 1$, which implies that B_p is not flat over R_p.

$(\subseteq)$:Suppose that B_p is not flat over R_p. Then $\wp = pB$ is a prime ideal of B such that $tr.deg_{\kappa(p)}\kappa(P) = 1$. Since A is integral over B, there exsits a prime ideal P of A such that $p = P \cap R$. Thus we have $tr.deg_{\kappa(p)}\kappa(P) = 1$.This implies that A_p is not flat over R_p.

(2)This follows from the integrality of A over B. $\square$

Theorem 3. *Let R be an integral domain with quotient field K and let K and let α be an anti-integral element of degree d over R. Let $A = R[\alpha]$ and let B be a subring of A containing R. Suppose that A is flat over R. If α is anti-integral element over B, then A is flat over B.*

proof. Note that $A = R[\alpha] = B[\alpha]$. Take $P \in Spec(A)$ and put $P \cap B = \wp$. Then $P \cap R = p$ is a prime ideal of R. Suppose that A is not flat over B. Then $tr.deg_{\kappa(\wp)}\kappa(P) > 0$. Since $\kappa(p) \subseteq \kappa(\wp) \subseteq \kappa(P)$, we have $tr.deg_{\kappa(p)}\kappa(P) > 0$. This contradicts that A is flat over R. $\square$

Finally, we give some results in integral extensions.

Proposition 4. *Let R be an integral domain with quotient field K and let $\alpha_i(1 \leq i \leq n)$ be anti-integral elements over R. Put $A = R[\alpha_1, \cdots, \alpha_n]$. Then the following statements are equivalent:*
(1) A is integral over R;
(2) $I_{[\alpha_i]} = R$ for all i.

proof.(1) $\Rightarrow$ (2): Since α_i are anti-integral and integral over R, $I_{[\alpha_i]} = R$ by [2.Theorem2.2].
(2) $\Rightarrow$ (1): If $I_{[\alpha_i]} = R$ for all i, then α_i are integral over R. Hence $A = R[\alpha_1, \cdots, \alpha_n]$ is integral over R. $\square$

Theorem5.*Let R be an integral domain with quotient field K and let $\alpha_i(1 \leq i \leq n)$ be anti-integral elements over R. Put $A = R[\alpha_1, \cdots, \alpha_n]$. Then, for any $p \in Spec(R)$, the following statements are equivalent:*
(1) A_p is integral over R_p;
(2) $p + \bigcap_{i=1}^{n} I_{[\alpha_i]}$.

proof.(2) $\Rightarrow$ (1): Since $p + I_{[\alpha_i]}$ for all i, $(I_{[\alpha_i]})_p = R_p$. Thus α_i are integral over R_p. Hence $A_p = R_p[\alpha_1, \cdots, \alpha_n]$ is integral over R_p.
(1) $\Rightarrow$ (2): If $A_p = R_p[\alpha_1, \cdots, \alpha_n]$ is integral over R_p, then α_i are anti-integral over R_p. Hence $(I_{[\alpha_i]})_p = R_p$ by Proposition 4. Therefore $p + \bigcap_{i=1}^{n} I_{[\alpha_i]}$. $\square$

References
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